Stokes' Theorem

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I. Introduction. Stokes' theorem on a manifold is a central theorem of mathematics. Special cases include the integral theorems of vector analysis and the Cauchy-Goursat theorem. My purpose here is to prove this version of

Stokes' Theorem. *Let ω be a continuous differential $(n-1)$-form on a compact oriented n-manifold M with boundary ∂M. Suppose that ω is differentiable on $M - \partial M$ and $d\omega$ is Lebesgue integrable there. Then*

$$\int_M d\omega = \int_{\partial M} \omega. \tag{1}$$

(ω is *differentiable* if its coefficient functions are differentiable.)

The recent development of multidimensional generalized Riemann integrals (see §II-D) makes it possible to prove strong versions of Stokes' theorem. Our version is a special case of a theorem of Pfeffer [22, Corollary 7.4], who uses his BV generalized Riemann integral. Unfortunately, his proof is quite involved.

The statement of the theorem and its proof here have several notable features. We describe them briefly and then elaborate in §II.

A. *The theorem requires only Lebesgue integrability of $d\omega$.* Most versions of Stokes' theorem require that $d\omega$ be continuous.

B. *The theorem requires $d\omega$ to exist only on $M - \partial M$.* Most versions of Stokes' theorem require $d\omega$ to exist on all of M.

C. *The proof uses the integral definition of $d\omega$.* The integral definition of $d\omega$ gives it a simple geometric meaning. The definition makes possible a simple and intuitive one line heuristic demonstration of Stokes' theorem on a cube, which shows us the *reason* for the theorem.

D. *The proof uses the Mawhin generalized Riemann integral.* This integral fits hand in glove with the integral definition of $d\omega$ to turn the heuristic demonstration of Stokes' theorem on a cube into a simple and intuitive proof on a cube.

Our proof of Stokes' theorem on a manifold proceeds in the usual two steps. First we prove the theorem for a cube. Here the proof is new and self contained. The statement and proof use the integral definition of $d\omega$ and the Mawhin integral. Then we lift the theorem from a cube to a manifold. Here we have nothing new to offer, but we give an outline of a standard proof for completeness. Along the way we must relate the integral definition of $d\omega$ to the usual definition using partial derivatives (see §V) and relate the Mawhin integral to the Lebesgue integral (see §VI). These more technical sections are the price we pay for features (A) – (D).

II. (A) – (D) Elaborated. **A.** *The theorem requires only Lebesgue integrability of $d\omega$.* Traditional proofs of Stokes' theorem, from those of Green's theorem on a rectangle to those of Stokes' theorem on a manifold, elementary and sophisticated alike, require that $\omega \in C^1$. See for example de Rham [5, p. 27], Grunsky [8, p. 97], Nevanlinna [19, p. 131], and Rudin [26, p. 272].

Stokes' theorem is a generalization of the fundamental theorem of calculus. Requiring $\omega \in C^1$ in Stokes' theorem corresponds to requiring f' to be continuous in the fundamental theorem of calculus. But an elementary proof of the fundamental theorem requires only that f' exist and be Riemann integrable on (a, b) (and that f be continuous on $[a, b]$): Let $a = x_0 < \ldots < x_j < \ldots < x_n = b$. Then using a telescoping series and the mean value theorem,

$$f(b) - f(a) = \sum_{j=1}^{n} \{f(x_j) - f(x_{j-1})\} = \sum_{j=1}^{n} f'(c_j)(x_j - x_{j-1}) \to \int_a^b f'. \qquad (2)$$

In fact, f' need only be Lebesgue integrable [25, Th. 8.21]. It is satisfying to have a version of Stokes' theorem which, like the fundamental theorem of calculus it generalizes, requires only Lebesgue integrability of the derivative.

We note that Acker's recent version of Stokes' theorem requires only that $d\omega$ be Riemann integrable [1]. This paper is well worth reading.

Standard versions of Green's theorem imply Cauchy's theorem: *If f is analytic with a continuous derivative in an open set containing a simple closed curve C and its interior, then $\int_C f(z)dz = 0$.* Our Stokes' theorem (and Acker's) specialize to a version of Green's theorem which implies the Cauchy-Goursat theorem: *If f is analytic in an open set containing a simple closed curve C and its interior, then $\int_C f(z)dz = 0$.* As Acker points out, this counters the usual view that the Cauchy-Goursat theorem is not a corollary of Green's theorem and so requires a special proof.

Our Stokes' theorem immediately yields Cauchy-Goursat's theorem on a manifold: *Let ω be an $(n-1)$-form continuous on M and differentiable on $M - \partial M$. Suppose that $d\omega \equiv 0$ on $M - \partial M$. Then $\int_{\partial M} \omega = 0$.* Using traditional versions of Stokes' theorem we would also need the hypothesis $\omega \in C^1$. This is the blemish Goursat's theorem removes from Cauchy's theorem.

B. *The theorem requires $d\omega$ to exist only on $M - \partial M$.* As seen above, the fundamental theorem of calculus requires $f\,'$ to exist only on the open interval (a, b). Again the situation with respect to Stokes' theorem is different: the references [5], [8], [19], and [26] cited above all require $d\omega$ to exist on all of M.

Since $d\omega$ need not exist on ∂M, in the Cauchy-Goursat theorem we need only assume that f is continuous on C and its interior, and analytic in the interior. This result can also be found in [13, Th. 3.10].

C. *The proof uses the integral definition of $d\omega$.* Let ω be an $(n-1)$-form on $\mathbf{R}^n$. Fix $x \in \mathbf{R}^n$. Let c denote an n-cube (of arbitrary orientation) with $x \in c$. Define

$$d\omega(x) \;=\; \lim_{\substack{x \in c \\ \mathrm{diam}(c) \to 0}} \frac{1}{|c|} \int_{\partial c} \omega \;. \tag{3}$$

(By a slight abuse of notation we identify the n-form $d\omega$ with its (single) coefficient function $d\omega(x)$: $d\omega = d\omega(x)\, dx_1 \wedge \ldots \wedge dx_n$.)

This *integral definition* gives $d\omega$ a clear geometric meaning.

The integral definition tells us the *reason* for Stokes' theorem. To see this, partition $[\,0,1]^n$ with small cubes $\{c_j\}$ and let $x_j \in c_j$. Then if $d\omega$ is Riemann integrable,

$$\int_{\partial[0,1]^n} \omega = \sum_j \int_{\partial c_j} \omega \approx \sum_j d\omega(x_j)\,|c_j| \to \int_{[0,1]^n} d\omega \;. \tag{4}$$

Note the step-by-step parallel between this heuristic argument and the proof of the fundamental theorem of calculus, Eq. (2).

The integral definition is essential in turning the heuristic argument into our proof of Stokes' theorem on a cube in §IV.

The integral definition does not refer to any coordinate system. In particular, $d\omega$ is invariant under a rotation of coordinates. In contrast, the usual *derivative definition* of $d\omega$ is given in terms of partial derivatives with respect to some coordinate system. It must then be proved that $d\omega$ is invariant under a rotation of coordinates.

In §V we show that if ω is differentiable, i.e., its coefficient functions are linearly approximable, then $d\omega$ exists and the integral definition is equivalent to the derivative definition.

One might say that the integral definition tells us *what $d\omega$ is*, whereas the derivative definition tells us *how to compute $d\omega$*.

For all these reasons, I prefer the integral definition of $d\omega$ to the derivative definition.

We can use the heuristic argument, Eq. (4), on a compact manifold with boundary M in $\mathbf{R}^n$ by "nearly" covering M with small "nearly" cubes. In this way, the divergence and Stokes' theorems of vector calculus can be made plausible.

The integral definition of $d\omega$ and the heuristic demonstration of Stokes' theorem are used in many physics oriented texts, e.g., [2, p. 188], [24, p. 10], [27, Sec. 5.8], and [30, pp. 83, 93]. They should be better known to mathematicians.

The definition and demonstration are used in the Harvard multivariable calculus text for the divergence theorem [9]. A teacher of multivariable calculus can be more comfortable with this approach knowing that it can be made rigorous.

D. *The proof uses the Mawhin generalized Riemann integral.* The first generalized Riemann integral was the *Henstock-Kurzweil integral* in $\mathbf{R}^1$ [10] [12]. Bartle has given an excellent elementary account of this integral [3].

The HK integral solves a problem in formulating the fundamental theorem of calculus: a derivative need not be Riemann, or even Lebesgue, integrable. A standard example is the function $f(x) = x^2 \cos(\pi/x^2)$ for $x \in (0, 1]$, with $f(0) = 0$. The derivative f' exists on $[0, 1]$. But f is not absolutely continuous on $[0, 1]$, and so f' is not Lebesgue integrable there. Among the impressive features of the HK integral is its formulation of the fundamental theorem:

If f' exists on $[a, b]$, then f' is HK-integrable on $[a, b]$ and

$$\text{(HK)} \int_a^b f'(x)dx = f(b) - f(a). \tag{5}$$

Equally impressive is the trivial proof of the theorem. All this even though f' need not be Lebesgue integrable. Moreover, the HK integral is *super Lebesgue*: If f is Lebesgue integrable, then it is HK integrable to the same value.

The HK integral in $\mathbf{R}^n$ is also super Lebesgue, but it does not always integrate $d\omega$, even if ω is differentiable [21, Example 5.7]. Mawhin designed his integral to overcome this deficiency [14, Theorem 2]. He proves this version of

Stokes' Theorem on a Cube. *Let ω be a differential $(n-1)$-form defined on an open set $U \supseteq [0, 1]^n$. If $d\omega$ exists on $[0, 1]^n$, then $d\omega$ is Mawhin integrable on $[0, 1]^n$ and*

$$\text{(M)} \int\limits_{[0,1]^n} d\omega = \int\limits_{\partial[0,1]^n} \omega \,. \tag{6}$$

Mawhin's proof uses his integral, but not the integral definition of $d\omega$. Acker proves his result using the integral definition, but not the Mawhin integral [1]. We give a proof of Eq. (6) in §IV which uses both; this provides a more general theorem than Acker's with a simpler and more intuitive proof than Mawhin's.

According to the theorem, the Mawhin integral always integrates $d\omega$. In addition, we shall show in §VI that the integral is super Lebesgue. Why, then, don't we abandon the Lebesgue integral in favor of the Mawhin integral? Most important for us, the change of variable theorem fails [22, p. 143], and so the integral cannot be lifted to manifolds. Fubini's theorem also fails [21, Remark 5.8]. And there are other deficiencies [21, Remark 7.3]. We use the Mawhin integral only as a catalyst to compute the Lebesgue integral on the left side of Eq. (1).

Pfeffer's BV integral *can* be lifted to manifolds. His version of Stokes' theorem is stated in terms of this integral: Under the hypotheses of our theorem, $d\omega$ is BV integrable and (BV) $\int_M d\omega = \int_{\partial M} \omega$ [22, Corollary 7.4]. Moreover, ω can be discontinuous on a "small" set [22, Theorem 7.3]. This is a strong result. But the BV integral is more complicated than the Mawhin integral, it does not fit well with the integral definition of $d\omega$, and the proof of Stokes' theorem is much more complicated than ours.

Unlike $\mathbf{R}^1$, where the HK integral seems to be completely satisfactory, none of the several generalized Riemann integrals in higher dimensions has enough desirable properties to make it a useful general purpose integral. Thus current versions of Stokes' theorem stated in terms of a generalized Riemann integral (e.g., [11], [15], [20], [22], [23]) cannot serve as a general purpose Stokes' theorem.

III. The Mawhin integral. We now give a series of definitions leading to the Mawhin integral [14] [15], specialized to $[0,1]^n$. (Mawhin calls it the RP – regular partition – integral.) For motivation and discussion of generalized Riemann integrals, see [3] and [16]. The definition of the Mawhin integral becomes that of the Riemann integral if the function $\delta(x)$ below is replaced with a constant δ and the cubes c_j with rectangles. We will see the definition in action in §4 and in §6.

A *gauge* on $[0,1]^n$ is a positive function $\delta(x)$ on $[0,1]^n$.

A *tagged regular partition* $\{c_j,\ x_j\}_{j=1}^k$ of $[0,1]^n$ is a decomposition of $[0,1]^n$ into closed subcubes $\{c_j\}$ together with points $x_j \in c_j$. The c_j are disjoint except for boundaries.

Let δ be a gauge on $[0,1]^n$. A tagged regular partition $\{c_j,\ x_j\}_{j=1}^k$ is *δ-fine* if $\operatorname{diam}(c_j) \le \delta(x_j),\ j = 1 \ldots k$.

Let $d\omega$ be an n-form defined on $[0,1]^n$. A number, denoted $(\mathrm{M})\int_{[0,1]^n} d\omega$, is the *Mawhin integral* of $d\omega$ over $[0,1]^n$ if, given $\varepsilon > 0$, there is a gauge δ on $[0,1]^n$ so that for every δ-fine tagged regular partition $\{c_j,\ x_j\}$ of $[0,1]^n$,

$$\left| (\mathrm{M}) \int_{[0,1]^n} d\omega - \sum_{j=1}^{k} d\omega(x_j)\,|\,c_j| \right| \le \varepsilon. \tag{7}$$

If this definition is to make sense, we need to prove two things:

(i) *Given a gauge δ on $[0,1]^n$, there is a δ-fine tagged regular partition of* $[0,1]^n$. (Cousin's lemma) To see this, first note that if a cube c is partitioned into subcubes, each of which has a δ-fine regular partition, then c has a δ-fine regular partition. Thus if $[0,1]^n$ has no δ-fine regular partition, then there is a sequence $[0,1]^n \supset c_1 \supset c_2 \supset \ldots$ of compact cubes with no δ-fine regular partition and diam $(c_i) \to 0$. Let $\{x\} = \bigcap_i c_i$. Choose j so that $\operatorname{diam}(c_j) \le \delta(x)$. Then $\{(c_j, x)\}$ is a δ-fine regular partition of c_j, which is a contradiction.

(It is interesting to note that the standard proof of the Cauchy-Goursat theorem and Acker's proof of Stokes' theorem use similar compactness arguments.)

(ii) *If the Mawhin integral exists, then it is unique.* For if δ_1 and δ_2 are gauges and $\delta = \mathrm{Min}(\delta_1, \delta_2)$, then a δ-fine regular partition is also δ_1-fine and δ_2-fine.

The use of tagged partitions is not limited to integration theory: Gordon has used them to prove many theorems of elementary real analysis [6].

IV. Proof of Stokes' theorem. We first prove the theorem on a cube, Eq. (6). Given $\varepsilon > 0$, define a gauge $\delta(x) > 0$ on $[0,1]^n$ as follows. Choose $x \in [0,1]^n$. Then according to the integral definition of $d\omega$, Eq. (3), there is a $\delta(x) > 0$ so that if $x \in c$, a cube with $\operatorname{diam}(c) \le \delta(x)$, then $\left| \int_{\partial c} \omega - d\omega(x)\,|c| \right| < \varepsilon |c|$. Now let $\{c_j, x_j\}$ be a δ-fine tagged regular partition of $[0,1]^n$. Then

$$\left| \int\limits_{\partial[0,1]^n} \omega \;-\; \sum_j d\omega(x_j)\,|c_j| \right| = \left| \sum_j \int\limits_{\partial c_j} \omega \;-\; \sum_j d\omega(x_j)\,|c_j| \right| < \sum_j \varepsilon\,|c_j| = \varepsilon.$$

By the definition of the Mawhin integral, Eq. (7), (M) $\int_{[0,1]^n} d\omega$ exists and is equal to $\int_{\partial[0,1]^n} \omega$. $\square$

As stated in §1, "The Mawhin integral fits hand in glove with the integral definition of $d\omega$ to turn the heuristic demonstration of Stokes' on a cube [Eq. (4)] into a simple and intuitive proof."

Corollary. *Let ω be a continuous differential $(n-1)$-form on $[0,1]^n$. Suppose that $d\omega$ exists on $(0,1)^n$ and is Lebesgue integrable there. Then*

$$\int\limits_{[0,1]^n} d\omega = \int\limits_{\partial[0,1]^n} \omega. \tag{8}$$

Proof. Let $c_k = \left[k^{-1}, 1-k^{-1}\right]^n$. From the result just proved and the fact that the Mawhin integral is super Lebesgue (see §VI) we have

$$\int\limits_{c_k} d\omega = \int\limits_{\partial c_k} \omega. \tag{9}$$

Let $k \to \infty$ in Eq. (9). The left side approaches the left side of Eq. (8) by the Lebesgue dominated convergence theorem. And the right side approaches the right side of Eq. (8) by the uniform continuity of ω on $[0,1]^n$. $\square$

We finish the proof of Stokes' theorem by lifting the corollary to a manifold. We give only an outline of a standard proof [18, pp. 303, 353], [28, p. 124], [29, p. 354], [2]. Filling in the details consists mostly of verifying that the concepts defined below are in fact well defined.

Choose $m \in M - \partial M$. Given a coordinate patch around m, translate and stretch its domain in $\mathbf{R}^n$ to obtain a coordinate patch $\varphi : U \to M$, where $U \supseteq [0,1]^n$ is open in $\mathbf{R}^n$, and $m \in \varphi((0,1)^n)$. Set $\varphi((0,1)^n) = V$. Similarly, for $m \in \partial M$, choose a coordinate patch $\varphi : U \to M$, where $U \supseteq [0,1]^n$ is open in the half space $\{x \in \mathbf{R}^n : x_n \ge 0\}$, and $m \in \varphi((0,1)^{n-1} \times 0)$. Set $\varphi((0,1)^{n-1} \times 0) = V$.

First suppose that the support of ω is contained in a V. According to §V, since $\varphi^*\omega$ is differentiable, $d(\varphi^*\omega)$ is given by the derivative definition Eq. (12), which is coordinate invariant. Thus $d\omega$ may defined in the usual way so that $\varphi^*(d\omega) = d(\varphi^*\omega)$. Then using the corollary (the outer equalities are definitions),

$$\int_M d\omega \;=\; \int_{[0,1]^n} \varphi^*(d\omega) \;=\; \int_{[0,1]^n} d\left(\varphi^*\omega\right) \;=\; \int_{\partial[0,1]^n} \varphi^*\omega \;=\; \int_{\partial M} \omega. \tag{10}$$

For a general ω, cover the compact manifold M with a finite number of open sets V_i of the type above. Let $\{f_i(x)\}$ be a partition of unity subordinate to $\{V_i\}$: $\operatorname{supp}(f_i) \subseteq V_i,\ f_i(x) \geq 0,\ f_i \in C^\infty$, and $\sum f_i(x) = 1$. Then $\omega = \sum f_i\,\omega$. Since $\operatorname{supp}(f_i\,\omega) \subseteq \operatorname{supp}(f_i) \subseteq V_i$, we may apply Eq. (10) to each $f_i\,\omega$:

$$\int_M d\omega = \sum_i \int_M d(f_i\,\omega) = \sum_i \int_{\partial M} f_i\,\omega = \int_{\partial M} \omega.$$

V. Existence of $d\omega$ and its coordinate representation. *Let*

$$\omega = \sum_{j=1}^n f_j(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \tag{11}$$

be an $(n-1)$-form, where the hat indicates that dx_j is omitted. If the f_j are differentiable at 0, then $d\omega(0)$ (defined by the integral definition) exists and is given by the derivative definition:

$$d\omega(0) = \sum_{j=1}^n (-1)^{j-1} \partial_j f_j(0). \tag{12}$$

Proof. By the integral definition of $d\omega$, Eq. (3), we must show that

$$\lim_{\substack{0\in c\\ \operatorname{diam}(c)\to 0}} \frac{1}{|c|} \int_{\partial c} \sum_{j=1}^n f_j(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_j} \wedge \ldots \wedge dx_n \;=\; \sum_{j=1}^n (-1)^{j-1} \partial_j f_j(0). \tag{13}$$

We first prove Eq. (13) for cubes with sides parallel to the x-axes. For such cubes it suffices to show that for an arbitrary p and differentiable function f,

$$\lim_{\substack{0\in c\\ \operatorname{diam}(c)\to 0}} \frac{1}{|c|} \int_{\partial c} f(x)\, dx_1 \wedge \ldots \wedge \widehat{dx_p} \wedge \ldots \wedge dx_n = (-1)^{p-1} \partial_p f(0). \tag{14}$$

Let c have width ε and sides $s_j^{\pm}$, on which x_j is constant. The only sides in ∂c contributing to the integral in Eq. (14) are $s_p^{\pm}$. And by definition, the orientation of $s_p^{\pm}$ in ∂c is $\pm(-1)^{p-1}$ times the orientation of $s_p^{\pm}$ in $\mathbf{R}^n$, i.e., $\pm(-1)^{p-1}(x_1, \ldots\, \widehat{x_p}\,, \ldots\, x_n)$ [28, p. 98]. Thus Eq. (14) can be written

$$\lim_{\varepsilon\to 0}\frac{(-1)^{p-1}}{\varepsilon^n}\left[\int_{s_p^+} f(x) - \int_{s_p^-} f(x)\right] = (-1)^{p-1}\partial_p f(0). \tag{15}$$

Our hypothesis that f is differentiable at 0 means that

$$f(x) = f(0) + \sum_{k=1}^{n} \partial_k f(0)\, x_k + R\,(x)\,, \tag{16}$$

where $|R(x)|\,/\,|x| \to 0$ as $|x| \to 0$.

We now prove Eq. (15) by substituting separately the three terms on the right side of Eq. (16) for $f(x)$ in the left side of Eq. (15). The result will be the right side of Eq. (15).

First term. Substitute $f(0)$ for $f(x)$ in the left side of Eq. (15). The two integrals are equal and so the result is zero.

Second term. For $x \in s_p^+$, let $\tilde{x} = (\tilde{x}_1, \ldots \tilde{x}_p, \ldots \tilde{x}_n)$ be the corresponding point on the opposite side s_p^-. Then $\tilde{x}_p = x_p - \varepsilon$, and for $k \neq p$, $\tilde{x}_k = x_k$. Substitute $\partial_k f(0)\, x_k$ for $f(x)$ in the left side of Eq. (15), omitting the limit:

$$\frac{(-1)^{p-1}\,\partial_k f(0)}{\varepsilon^n}\int_{s_p^+}(x_k - \tilde{x}_k).$$

If $k = p$, this expression is equal to the right side of Eq. (15). If $k \neq p$, the expression is zero.

Third term. Substitute $R\,(x)$ for $f(x)$ in the left side of Eq. (15). Since $|x| \leq \sqrt{n}\varepsilon$ on c,

$$\left|\frac{(-1)^{\,p-1}}{\varepsilon^n}\int_{s_p^\pm} R(x)\right| \;\leq\; \frac{1}{\varepsilon^n}\int_{s_p^\pm}\frac{\sqrt{n}\,\varepsilon}{|x|}\,|R(x)| \;\leq\; \sqrt{n}\sup_{|x|\,\leq\,\sqrt{n}\varepsilon}\frac{|R(x)|}{|x|} \to\; 0. \tag{17}$$

We have now proved Eq. (13) for cubes with sides parallel to the axes. However, the limit in Eq. (13) is taken as $\mathrm{diam}(c) \to 0$ for cubes of arbitrary orientation. Thus it remains to show that the limit is *independent of* and *uniform in*, the orientation of the cubes.

The only limit taken in proving Eq. (15) is in Eq. (17). This limit is independent of and uniform in the orientation of the cubes because $R(x)$ is invariant under a rotation of coordinates. To see this, observe that the other three terms in Eq. (16) are invariant under a rotation. ($f(x)$ is independent of the coordinates assigned to the point x, and the sum is $\nabla f \cdot \mathbf{x}$, where, since f is differentiable, ∇f is a vector.)

VI. The Mawhin integral is super Lebesgue. *If f is Lebesgue integrable on $[0,1]^n$, then it is Mawhin integrable on $[0,1]^n$ to the same value.*

Proof. (From [4].) Let $\epsilon > 0$ be given. Choose η, $0 < \eta < \epsilon$, so that if $\mu(a) < \eta$ then $\int_a |f| < \epsilon$.

For each integer i, set

$$e_i = \{x \in [0,1]^n : i\epsilon < f(x) \le (i+1)\epsilon\}.$$

The measurable and disjoint sets e_i cover $[0,1]^n$.

For each integer i choose an open set $g_i \supseteq e_i$ with

$$\mu(g_i - e_i) < \frac{\eta}{3 \cdot 2^{|i|}(|i|+1)}.$$

Define a gauge δ: for $x \in e_i$ set $\delta(x) = \operatorname{dist}(x,\ \tilde{g}_i)$. ($\tilde{g}_i = [0,1]^n - g_i$).

Let $\{c_j,\ x_j\}_{j=1}^k$ be a δ-fine tagged regular partition of $[\,0,1]^n$.

Now let n_j be the integer for which $x_j \in e_{n_j}$. Decompose c_j into $a_j = c_j \cap e_{n_j}$ and $b_j = c_j - e_{n_j}$. Then

$$\left| \int_{[0,1]^n} f - \sum_{j=1}^k f(x_j)|c_j| \right| = \left| \sum_{j=1}^k \int_{c_j} (f(t) - f(x_j))dt \right|$$

$$\le \sum_{j=1}^k \int_{a_j} |f(t) - f(x_j)|dt + \sum_{j=1}^k \int_{b_j} |f| + \sum_{j=1}^k \int_{b_j} |f(x_j)|dt\ .$$

We finish the proof by showing that each of the three terms on the right is $\le \epsilon$.

First term. In each integral, $t \in a_j \subseteq e_{n_j}$ and $x_j \in e_{n_j}$. Thus $|f(t) - f(x_j)| < \epsilon$. Moreover, since $a_j \subseteq c_j$ and the c_j are a.e. disjoint, the a_j are a.e. disjoint. Thus

$$\sum_{j=1}^k \int_{a_j} |f(t) - f(x_j)|dt \le \int_{[0,1]^n} \epsilon = \epsilon\ .$$

Second term. As with the a_j, the b_j are a.e. disjoint. Since $\{c_j,\ x_j\}_{j=1}^k$ is δ-fine, $\operatorname{diam}(c_j) \le \delta(x_j) = \operatorname{dist}\big(x_j,\ \tilde{g}_{n_j}\big)$. Thus $c_j \subseteq g_{n_j}$. Subtract e_{n_j} from this expression, giving $b_j \subseteq g_{n_j} - e_{n_j}$. It follows that the (a.e. disjoint) union of all those b_js with the same n_j is contained in $g_{n_j} - e_{n_j}$. Thus

$$\sum_{j=1}^k \mu(b_j) \le \sum_{n_j} \mu(g_{n_j} - e_{n_j}) < \sum_{n_j} \frac{\eta}{3 \cdot 2^{|n_j|}} < \eta,$$

whence $\sum_j \int_{b_j} |f| < \epsilon$ by the definition of η.

Third term. Since $\eta < \epsilon$ we have

$$\sum_{j=1}^k \int_{b_j} |f(x_j)|dt = \sum_{j=1}^k |f(x_j)|\mu(b_j) < \sum_{n_j} (|n_j|+1)\frac{\eta}{3 \cdot 2^{|n_j|}(|n_j|+1)} < \epsilon.$$

The McShane integral. The Lebesgue integral can be formulated as a generalized Riemann integral, called the *McShane integral* [17], [7]. Its definition is the same as that of the Mawhin integral, except that more partitions must satisfy Eq. (7) than the δ-fine tagged regular partitions of the Mawhin integral. This implies that the Mawhin integral is super McShane.

First, allow rectangles as well as cubes in a partition. This gives the multi-dimensional HK integral.

Next, replace the condition $\operatorname{diam}(c_j) \le \delta(x_j)$ in the definition of a δ-fine tagged partition with the condition $c_j \subseteq B\,(x_j, \delta(x_j))$, the ball centered at x_j with radius $\delta(x_j)$. Given that $x_j \in c_j$, the two conditions are equivalent. Now drop the requirement that $x_j \in c_j$. (The condition $c_j \subseteq B\,(x_j, \delta(x_j))$ keeps x_j close to c_j.) This gives the McShane integral.

The proof above that the Mawhin integral is super Lebesgue also shows that the McShane integral is super Lebesgue. (The only thing to check is that $c_j \subseteq g_{n_j}$. This follows from $c_j \subseteq B\,(x_j, \delta(x_j))$.) In fact, the Lebesgue and McShane integrals are equivalent [17, p. 296]. McShane develops the Lebesgue integral using his generalized Riemann integral definition.

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