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Applications of Clifford ratios unaffected by the local Schwarz paradox

Paolo Roselli

Dipartimeno di Matematica, Università di Roma "Tor Vergata", Rome, Italy

Correspondence
Paolo Roselli, Dipartimeno di Matematica, Università di Roma "Tor Vergata", Rome, Italy.
Email: roselli@mat.uniroma2.it

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We show that the gradient of a strongly differentiable function at a point is the limit of a single coordinate-free Clifford quotient between a multidifference pseudo-vector and a pseudo-scalar, or of a sum of Clifford quotients between scalars (as numerators) and vectors (as denominators), both evaluated at the vertices of a same non-degenerate simplex contracting to that point. Such result allows to fix an issue with a defective definition of pseudo-scalar field in Sobczyck's simplicial calculus. Then, we provide some consequences and conjectures implied by the foregoing results.

KEYWORDS

Clifford algebras, computational geometry, difference quotient, geometric algebra, gradient, pseudoscalar field, simplicial calculus

MSC CLASSIFICATION

53A45, 15A67, 53A70, 57-08

1 | INTRODUCTION

The length of a smooth curve in a Euclidean space is the limit of the lengths of polygonal lines with rectilinear segments whose endpoints, on the curve, uniformly[1] converge. An analog convergence result does not hold for the area of a smooth curved surface. This unexpected phenomenon is usually called "surface area paradox," or "Schwarz paradox" (see, e.g., [1–5] and [6]).

The Schwarz paradox is rooted in the following local divergence result that we call "local Schwarz paradox":

> *the plane, secant a curved surface at three non-collinear points approaching a point* $\mathbf{x}_0$ *of the surface, does not necessarily approach the tangent plane of the surface at* $\mathbf{x}_0$ (see Section 1.2 for details).

Both the classical and the local Schwarz paradoxes are sometimes overlooked. For instance, Macdonald has noticed in [6] that Sobczyk's definition[2] of the pseudoscalar tangent field to a surface is affected by the local Schwarz paradox. As a matter of fact, in the context of Sobczyk's simplicial calculus [7], the local Schwarz paradox can be rephrased as follows:

> *the normalized 2-blades of a sequence of non-degenerate triangles (2-simplexes), with vertices lying on a smooth orientable surface converging to a point* $\mathbf{x}_0$*, do not necessarily converge to the unit blade* $I(\mathbf{x}_0)$*, tangent to the surface at* $\mathbf{x}_0$.

[1]The length of each segment goes to 0 uniformly.
[2]See relation (3.4) at page 6 of [7].

If the surface is the graph of a bivariate smooth nonlinear function f, the local Schwarz paradox appears as follows[3]:

> *the vector* $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ *describing the plane* $z = f(\mathbf{a}) + \mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} \cdot (\mathbf{x} - \mathbf{a})$, *secant the graph of* f *at the three non-collinear points* $(\mathbf{a}, f(\mathbf{a}))$, $(\mathbf{b}, f(\mathbf{b}))$, *and* $(\mathbf{c}, f(\mathbf{c}))$, *does not necessarily converge to the gradient* $\nabla f(\mathbf{x}_0)$, *as* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *converge to* $\mathbf{x}_0$

In this work, we explicitly construct a new vector $\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ converging to the gradient $\nabla f(\mathbf{x}_0)$, as $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ converge to $\mathbf{x}_0$. So, the corresponding plane $z = f(\mathbf{a}) + \bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} \cdot (\mathbf{x} - \mathbf{a})$ does converge to the plane $z = f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0})$ tangent the graph of f at point $(\mathbf{x}_0, f(\mathbf{x}_0))$, regardless how the three non-collinear points converge to $\mathbf{x}_0$. Moreover, we show that $\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ can be expressed as a Clifford ratio in two evocative forms (see Section 3.3.2 and Section 3.4 for details):

- $\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \overline{\Delta} f / \Delta$, where
  - $\overline{\Delta} f$ is the vector $\frac{1}{2}\{[f(\bar{\mathbf{a}}) - f(\mathbf{a})](\mathbf{c} - \mathbf{b}) - [f(\mathbf{c}) - f(\mathbf{b})](\bar{\mathbf{a}} - \mathbf{a})\}$, and
  - Δ is the 2-blade $(\mathbf{a} \wedge \mathbf{b}) + (\mathbf{b} \wedge \mathbf{c}) + (\mathbf{c} \wedge \mathbf{a}) = (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})$;
- $\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{f(\bar{\mathbf{a}}) - f(\mathbf{a})}{\bar{\mathbf{a}} - \mathbf{a}} + \frac{f(\mathbf{c}) - f(\mathbf{b})}{\mathbf{c} - \mathbf{b}}$; that is, a ratio between numbers (as numerators) and vectors (as denominators).

Then, we extend[4] such main results[5] to approximate gradients of general multivariate smooth functions.

As the pseudo-scalar field $I(\mathbf{x}_0)$, tangent to a k-surface φ_k, can be expressed[6] in a local chart[7] $\mathbf{x} : \Omega \subseteq \mathbb{E}_k \to \varphi_k$ by gradients

$$I(\mathbf{x}_0) = \frac{\sum\limits_{1 \le i_1 < \cdots < i_k \le n} \left[\left(\nabla \chi_{i_1(\mathbf{u}_0)} \wedge \cdots \wedge \nabla \chi_{i_k(\mathbf{u}_0)} \right) I_k^{-1} \right] \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k}}{\left| \sum\limits_{1 \le i_1 < \cdots < i_k \le n} \left[\left(\nabla \chi_{i_1(\mathbf{u}_0)} \wedge \cdots \wedge \nabla \chi_{i_k(\mathbf{u}_0)} \right) I_k^{-1} \right] \mathbf{e}_{i_1} \wedge \cdots \wedge \mathbf{e}_{i_k} \right|}$$

(where $\mathbf{x} = \mathbf{x}_{(\mathbf{u})} = \chi_{1(\mathbf{u})}\mathbf{e}_1 + \ldots + \chi_{n(\mathbf{u})}\mathbf{e}_n$ is the orthonormal decomposition in the n-dimensional Euclidean space $\mathbb{E}_n$ of vector $\mathbf{x}$, parametrizing locally the k-surface, $\mathbf{x}_0 = \mathbf{x}_{(\mathbf{u}_0)}$, $\mathbf{u}_0$ is a point internal to Ω, and I_k is a pseudo-unit, or "orientation," of $\mathbb{E}_k$), our results allow to define $I(\mathbf{x}_0)$ as the limit of suitable normalized k-blades based on non-degenerate k-simplexes contracting to $\mathbf{x}_0$.

Another possible method to give a proper definition of the pseudo-scalar field, not using charts, could be to select suitable sequences of k-chains in $\mathbb{E}_n$ converging to the k-surface ϕ_k. However, here we do not have enough room to prove such last claim. So we have to postpone its detailed statement and proof to a further work.

1.1 | Further implications

Here, we sketch some other possible applications of the foregoing results in pure and applied mathematics.

1.1.1 | Leibniz's notation

At page 45 of [9], the following is written:

> *Leibniz's notation* $dF/d\tau$ *or* $\partial F/\partial\tau$ *emphasizes the definition of derivative as the limit of a difference quotient. It will be seen that differentiation by a general multivector cannot be defined by a difference quotient, so Leibniz's notation is appropriate only for scalar variables.*

In this work, we show[8] that, using as product the Clifford geometric product, as numerator our mean multidifference pseudo-vector $\overline{\Delta} f$, and as denominator the pseudo-scalar Δ, Leibniz's quotient notation

$$\overline{\Delta} f / \Delta = \left(\overline{\Delta} f_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} \right) \left(\Delta_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} \right)^{-1}$$

[3]See Section 1.3 for details and [8] for further geometric results.
[4]See Theorem 2 and Theorem 3.
[5]Theorem 1.
[6]Here, we adopt some notations taken from [7].
[7]That is, a local regular parametrization.
[8]See Section 3.4, Section 4.4, and Section 5.4.

remains appropriate to approximate (or define) the gradient of multivariable scalar functions (provided they are strongly differentiable).

1.1.2 | Fundamental theorem of calculus and nonabsolute change of variable formula

The foregoing observation suggests to use the new Leibniz's difference quotient (that should be properly called "Clifford multi-difference quotient") as a tool also for higher dimensional Calculus. If this is the case, many paths are open to explore classical topics. For example, we may try to extend to higher dimensions the elegant definition[9] of nonabsolute integration given by Kurzweil and Henstock by directly using the new Clifford multidifference quotient. In particular, we guess the possibility of defining a multivector valued generalized Riemann integral on orientable hypervolumes and hypersurfaces such that a fundamental theorem of calculus of the following form:

$$\underbrace{\iint}_{\substack{\text{oriented}\\ \text{hyper-volume}\\ \Omega}} \underbrace{\underbrace{\nabla f(\mathbf{x})}_{\text{vector}}\ \underbrace{d\mathbf{x}}_{\text{pseudo-scalar}}}_{\text{pseudo-vector}} = \underbrace{\int}_{\substack{\text{oriented}\\ \text{rectifiable}\\ \text{hyper-surface}\\ \partial\Omega}} \underbrace{\underbrace{f(\mathbf{x})}_{\text{scalar}}\ \underbrace{d\mathbf{x}}_{\text{pseudo-vector}}}_{\text{pseudo-vector}} \tag{1.1}$$

may hold, for an everywhere strongly differentiable function $f : \Omega \subset \mathbb{E}_n \rightarrow \mathbb{R}$. One could argue that this strong regularity condition may limit the range of possible results. Despite this possible limitation, it would be interesting to explore more deeply a fundamental theorem of calculus in a form[10] different from the "divergence" one. Moreover, once defined the generalized integrals satisfying (1.1), one could obtain a subsequent non absolute change of variable formula for multiple integrals not involving the absolute value of the Jacobian determinant, as one would expect by analogy with the change of variable formula for 1-dimensional integrals.

1.1.3 | Finite element method

Delaunay triangulations maximize the minimum angle of all angles of the triangles in a triangulation of a plane point cloud. This extremal property allows in the finite element method analysis to prevent divergence phenomena due to the Schwarz paradox. The algorithms we have used do not suffer such divergence phenomena; so, they could be used in the finite element method on arbitrary triangulations (non-necessarily of Delaunay type).

One can profit further of the above possibility to neglect Delaunay triangulations. In fact, for n-dimensional point clouds (with $n \geq 3$), it is increasingly difficult to generate a Delaunay "tretrahedralization." On the contrary, we claim that our convergent algorithms (independent of the maximal Delaunay property) can be applied to efficiently discretize differential operators on any n-dimensional smooth manifold.

1.1.4 | Teaching

The classical global Schwarz paradox deals with the area of surfaces, and its presentation is often omitted from advanced calculus curricula. In this work, we have shown that the local form of the Schwarz paradox involves the very definition of gradient and could be presented at the beginning of a multivariable differential calculus course.[11] In this sense, it provides further motivation to introduce Clifford geometric algebra as a unifying language for Geometry and Analysis.[12]

1.2 | The local Schwarz paradox

The derivative $f'(x_0)$ of a single-variable scalar function $f : \Omega \subseteq \mathbb{R} \rightarrow \mathbb{R}$ at a point x_0 internal to Ω is the limit of the quotient $\frac{\Delta f_{(a,x_0)}}{\Delta_{(a,x_0)}}$, between the differences $\Delta f_{(a,x_0)} = f(a) - f(x_0)$, and $\Delta_{(a,x_0)} = a - x_0$

$$\lim_{a\rightarrow x_0} \frac{\Delta f_{(a,x_0)}}{\Delta_{(a,x_0)}} = f'(x_0)$$

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[9]See [10].
[10]See, for instance, [11], although it is affected by the Schwarz paradox.
[11]As proposed in [8], for instance.
[12]As advocated in [12] and [13].

The strong derivative $f^*(x_0)$ is defined[13] by a stronger, but fully symmetric, limit

$$\lim_{\substack{(a,b)\to(x_0,x_0)\\ a,b \text{ distinct}}} \frac{\Delta f_{(a,b)}}{\Delta_{(a,b)}} = f^*(x_0)$$

of the difference quotient $\frac{f(a)-f(b)}{a-b}$. So, $f^*(x_0) = f'(x_0)$ when the strong derivative exists. The existence of those limits corresponds to a well-known geometric phenomenon: The line secant the graph of f at points $(a, f(a))$, $(b, f(b))$

$$y = f(a) + \frac{\Delta f_{(a,b)}}{\Delta_{(a,b)}}(x-a) \tag{1.2}$$

assumes, as the non-degenerate segment joining a and b contracts to x_0, the limit position

$$y = f(x_0) + f'(x_0)(x - x_0) \tag{1.3}$$

which is the line tangent the graph of f at point $(x_0, f(x_0))$.

By analogy, one would expect that planes secant the graph of a two-variable real function at three non-collinear points always assume as limit position (as those three points converge on the graph to a same limit point) the position of the plane tangent the graph at that limit point. Amazingly, this is not the case, even for smooth functions.[14]

Example 1. Let $f(x, y) = \sqrt{1-x^2}$. The plane secant the graph $\{(x, y, f(x, y)) : (x, y) \in [-1, 1] \times \mathbb{R}\}$ at points $(0, 0, 1)$, $\left(-\alpha, \beta, \sqrt{1-\alpha^2}\right)$, and $\left(\alpha, \beta, \sqrt{1-\alpha^2}\right)$ is defined by the relation between $(x, y, z) \in \mathbb{R}^3$

$$z = 1 - \frac{\alpha^2}{\beta\left(1+\sqrt{1-\alpha^2}\right)} y \tag{1.4}$$

- If $\beta = \alpha$ and $\alpha \to 0$, then the limit position of the secant plane (1.4) is the plane $z = 1$, which is tangent to the graph of f at $(0, 0, 1)$;
- If $\beta = \alpha^2$ and $\alpha \to 0$, then the limit position of the secant plane (1.4) is the plane $z = 1 - \frac{1}{2}y$, which is not tangent to the graph of f at point $(0, 0, 1)$;
- If $\beta = \alpha^3$ and $\alpha \to 0$, then the limit position of the secant plane (1.4) is the plane $y = 0$, which is even orthogonal to the tangent plane!

1.3 | Some notations for the following

In this work, we perform coordinate-free vector computations. In order to better distinguish dimensionless numbers (i.e., scalars) from vectors, we adopt the following notations:

- vectors are denoted by **bold** Latin lower case letters;
- real numbers are denoted by nonbold Latin or Greek lower case letters;
- $\mathbb{E}_n$ denotes a n-dimensional Euclidean space: a real vector space with a positive definite symmetric bilinear form; this bilinear form is denoted by $\mathbf{u} \cdot \mathbf{v}$, for each $\mathbf{u}, \mathbf{v} \in \mathbb{E}_n$.

In particular, if $\{\mathbf{e}_1, \mathbf{e}_2\}$ is an orthonormal basis for $\mathbb{E}_2$ (i.e., $\mathbf{e}_1 \cdot \mathbf{e}_2 = 0$, and $\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = 1$), $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2$, $\mathbf{a} = \mathbf{0}$, $\mathbf{b} = -\alpha\mathbf{e}_1 + \beta\mathbf{e}_2$, $\mathbf{c} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$, then the foregoing Example 1 can be resumed as follows: The secant plane $z = f(\mathbf{a}) - \frac{\alpha^2}{\beta\left(1+\sqrt{1-\alpha^2}\right)} y$ has no limit position when the non-degenerate triangle (having vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$) contracts to the point $\mathbf{0}$, because $\lim\limits_{(\alpha,\beta)\to(0,0)} \frac{\alpha^2}{\beta\left(1+\sqrt{1-\alpha^2}\right)}$ does not exist.

[13] See, for example, [14–16].
[14] See also [8].

In general, for a nonlinear two-variable function f, the local Schwarz paradox corresponds to the nonexistence of the strong limit

$$\lim_{\substack{(\mathbf{a},\mathbf{b},\mathbf{c})\to(\mathbf{x_0},\mathbf{x_0},\mathbf{x_0}) \\ \mathbf{a},\mathbf{b},\mathbf{c}\ \text{not collinear}}} \mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$$

where $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ is the vector in $\mathbb{E}_2$ such that the secant plane to the graph of f at points $(\mathbf{a}, f(\mathbf{a}))$, $(\mathbf{b}, f(\mathbf{b}))$, and $(\mathbf{c}, f(\mathbf{c}))$ has Cartesian equation in $\mathbb{E}_2 \oplus \mathbb{R}$

$$z = f(\mathbf{a}) + \mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} \cdot (\mathbf{x} - \mathbf{a}), \tag{1.5}$$

when $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vertices of a non-degenerate triangle internal to the domain Ω of f.

Remark 1. The letter “r” in $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ stands for “ratio,” because we will show that such vector can be considered as a ratio of a pseudo-vector and a pseudo-scalar in a suitable Clifford algebra.

Remark 2. Vectors of $\mathbb{E}_n$ will also be called “points,” because we identify the Euclidean vector space $\mathbb{E}_n$ with a Euclidean affine[15] space $\mathcal{E}_n$ (modeled on $\mathbb{E}_n$) where an arbitrary point $O \in \mathcal{E}_n$ is considered as reference point, and it is identified with the zero vector $\mathbf{0} \in \mathbb{E}_n$. This allows us to interpret geometrically some subsets of $\mathbb{E}_n$. For instance,

- a “line” in $\mathbb{E}_n$ is the set

$$\mathcal{L}_{(\mathbf{a},\mathbf{b})} = \{\alpha\mathbf{a} + \beta\mathbf{b} \;:\; \alpha, \beta \in \mathbb{R}, \alpha + \beta = 1\}$$

  for some distinct points $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{E}_n$;
- three distinct points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{E}_n$ are “collinear” if they all belong to a same line;
- a “plane” in $\mathbb{E}_n$ is the set

$$\mathcal{L}_{(\mathbf{a},\mathbf{b},\mathbf{c})} = \{\alpha\mathbf{a} + \beta\mathbf{b} + \gamma\mathbf{c} \;:\; \alpha, \beta, \gamma \in \mathbb{R}, \alpha + \beta + \gamma = 1\}$$

  for some distinct and non-collinear points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{E}_n$;
- four distinct points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in $\mathbb{E}_n$ are “coplanar” if they all belong to a same plane.

1.4 | Summary of this work

In this work, we first provide explicit coordinate-free expressions of vector $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} \in \mathbb{E}_2$, both

- as a linear combination[16] of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ (in Section 3.2) and,
- in Section 3.1, as a noncommutative quotient

$$\left(\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}$$

  of a multidifference vector

$$\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = [f(\mathbf{b}) - f(\mathbf{a})]\,(\mathbf{c} - \mathbf{a}) - [f(\mathbf{c}) - f(\mathbf{a})]\,(\mathbf{b} - \mathbf{a}) \in \mathbb{E}_2$$

  and the oriented area

$$\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = (\mathbf{a} \wedge \mathbf{b}) + (\mathbf{b} \wedge \mathbf{c}) + (\mathbf{c} \wedge \mathbf{a}) = (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \in \mathbb{G}_{\binom{2}{2}}$$

  with respect to the geometric product of the Clifford algebra[17]

$$\mathbb{G}_2 = \mathcal{C}\ell_{(\mathbb{E}_2)} = \mathcal{C}\ell_{2,0} \simeq \mathbb{R} \oplus \mathbb{E}_2 \oplus \mathbb{G}_{\binom{2}{2}}$$

[15] See for instance [17].
[16] Another linear combination, using vectors normal to the sides of the triangle determined by the points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, can be found in [8].
[17] See, for example, [9, 18–22], and Section 4.3.1 (Remark 17) for the notation $\mathbb{G}_{\binom{n}{k}}$.

generated by the 2-dimensional Euclidean space $\mathbb{E}_2$.

Then, in Section 3.4, we introduce a new multidifference vector $\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})}$, such that the corresponding plane

$$z = f(\mathbf{a}) + \left[\left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} \right) \left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} \right)^{-1} \right] \cdot (\mathbf{x} - \mathbf{a})$$

(called "mean secant plane") always assumes, as limit position, that of the tangent plane

$$z = f(\mathbf{x_0}) + \nabla f(\mathbf{x_0}) \cdot (\mathbf{x} - \mathbf{x_0}) \tag{1.6}$$

because we prove[18] that the limit

$$\lim_{\substack{(\mathbf{a},\mathbf{b},\mathbf{c}) \to (\mathbf{x_0},\mathbf{x_0},\mathbf{x_0}) \\ \mathbf{a},\mathbf{b},\mathbf{c} \text{ not collinear}}} \left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} \right) \left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} \right)^{-1}$$

always exists when f is strongly differentiable[19] at $\mathbf{x}_0$, and it is equal to the gradient $\nabla f(\mathbf{x}_0)$. Moreover, we show in Proposition 1 that the vector $\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$, corresponding to the Clifford ratio $\left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} \right) \left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} \right)^{-1}$, can always be written as a sum of quotients between numbers (as numerators) and vectors (as denominators), strongly resembling the scalar difference quotients; more precisely, we will prove that

$$\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{f(\bar{\mathbf{a}}) - f(\mathbf{a})}{\bar{\mathbf{a}} - \mathbf{a}} + \frac{f(\mathbf{c}) - f(\mathbf{b})}{\mathbf{c} - \mathbf{b}}$$

where $\bar{\mathbf{a}}$ is uniquely defined by points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in Section 3.3.2.

Then, in Section 4, we extend the foregoing results to dimension three and in Section 5 to arbitrary higher dimensions.[20] We decided to proceed to the latter general case slowly, because a reader eventually concerned by the local Schwarz paradox is not necessarily acquainted to general formulas in geometric algebra, and we esteem that the low dimensional cases involve computations more affordable for freshmen to Clifford algebras than general relations in Geometric Algebra as presented in Section 5.

Remark 3. While the tangent and secant lines are represented by analog scalar equations such as (1.2) and (1.3), tangent and secant planes are proposed differently: The equation of the tangent plane is universally known in the coordinate-free vector form (1.6), but we have never met in the literature the equation of the secant plane in the coordinate-free vector form (1.5). The lack of vector $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ from calculus texts makes difficult even asking the question of which could be the vector analogy of the scalar difference quotient. This gap is not surprising, since vector $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ is a Clifford ratio and most mathematicians are unfamiliar with geometric algebra.

2 | SOME REMINDERS OF GEOMETRIC ALGEBRA

As announced, in this work, we use the associative vector algebra

$$\mathbb{G}_n = C\ell_{(\mathbb{E}_n)} = C\ell_{n,0}$$

which is the Clifford geometric algebra[21] generated by the n-dimensional Euclidean space $\mathbb{E}_n$ (the geometric product being denoted by juxtaposition). In particular, in $\mathbb{G}_n$, we have that

- for all $\mathbf{v} \in \mathbb{E}_n$ $\mathbf{v}\mathbf{v} = \mathbf{v}^2 = \mathbf{v} \cdot \mathbf{v}$, which implies that $\frac{1}{2}(\mathbf{u}\mathbf{v} + \mathbf{v}\mathbf{u}) = \mathbf{u} \cdot \mathbf{v}$, for all $\mathbf{u}, \mathbf{v} \in \mathbb{E}_n$;
- scalars always commute with the geometric product;

[18]see Theorem 1.
[19]See Definition 1.
[20]See Theorems 2 and 3.
[21]See, for example, [9, 18–22].

- if $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{E}_n$ (i.e., $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ when $i \neq j$, and $\mathbf{e}_i \cdot \mathbf{e}_i = 1$), then

$$\{1\} \cup \left\{\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_k}\right\}_{1 \le i_1 < \cdots < i_k \le n} \quad \text{with} \quad 1 \le k \le n$$

  is a basis for $\mathbb{G}_n$. So, the associative vector algebra $\mathbb{G}_n$ has dimension 2^n.

Thus, for instance,

$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_1\mathbf{e}_2\} \text{ is a basis for } \mathbb{G}_2;$$
$$\{1, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\} \text{ is a basis for } \mathbb{G}_3$$

Defining $\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu})$ for each $\mathbf{u}, \mathbf{v} \in \mathbb{E}_n$, you can verify that $\mathbf{u} \wedge \mathbf{v} = -\mathbf{v} \wedge \mathbf{u}$, $\mathbf{u} \wedge \mathbf{u} = 0$, and $\mathbf{u} \wedge \mathbf{v} = \mathbf{uv}$ when vectors $\mathbf{u}$ and $\mathbf{v}$ are mutually orthogonal (i.e., $\mathbf{u} \cdot \mathbf{v} = 0$).

Remark 4. Note that, for each $\mathbf{u}, \mathbf{v} \in \mathbb{E}_n$, we have that $\mathbf{uv} = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \wedge \mathbf{v})$. In what follows, to limit the use of parentheses, we adopt the following rule of precedence between operations: geometric product is used first, secondly the scalar product "$\cdot$," then "$\wedge$" is performed, and finally the sum "+."

2.1 | The 2×2 determinant as a Clifford quotient and as a scalar product

Clifford geometric algebra allows coordinate-free vector computations having interesting geometric interpretations. For example, we can give a coordinate-free interpretation to the determinant of a 2×2 real matrix

$$\begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix}$$

as a Clifford ratio. As a matter of fact, let us fix any ordered couple $\mathbf{e}_1$, $\mathbf{e}_2$ of orthonormal vectors in $\mathbb{E}_n$ (with $n \ge 2$), and let us consider $\mathbf{u} = \mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2$ and $\mathbf{v} = \nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2 \in \mathbb{E}_2 \subseteq \mathbb{E}_n$, then

$$\begin{aligned}(\mathbf{u} \wedge \mathbf{v})(I_2)^{-1} &= [(\mu_1\mathbf{e}_1 + \mu_2\mathbf{e}_2) \wedge (\nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2)]\,(\mathbf{e}_1\mathbf{e}_2)^{-1} \\ &= \det\begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} \mathbf{e}_1\mathbf{e}_2(\mathbf{e}_2\mathbf{e}_1) = \det\begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix}\end{aligned}$$

as $I_2 = \mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$, and $(I_2)^{-1} = \mathbf{e}_2\mathbf{e}_1 = -\mathbf{e}_1\mathbf{e}_2 = -I_2$. Note that, as $(\mathbf{u} \wedge \mathbf{v})(I_2)^{-1} = (I_2)^{-1}(\mathbf{u} \wedge \mathbf{v})$, one could also write $\det\begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} = \frac{\mathbf{u} \wedge \mathbf{v}}{I_2}$.

Remark 5. The geometric product I_2 does not depend on the particular orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ of $span\{\mathbf{e}_1, \mathbf{e}_2\}$, but only on its orientation. More precisely, if $\{\mathbf{g}_1, \mathbf{g}_2\}$ is any other orthonormal basis of $span\{\mathbf{e}_1, \mathbf{e}_2\}$, then $\mathbf{g}_1\mathbf{g}_2 (= \mathbf{g}_1 \wedge \mathbf{g}_2)$ is equal to I_2 or $-I_2$. That is why I_2 is called an "orientation" of $span\{\mathbf{e}_1, \mathbf{e}_2\}$. An analog property and definition is valid for the product $\mathbf{e}_1 \cdots \mathbf{e}_k = I_k$ of any ordered list of mutually orthonormal vectors in $\mathbb{E}_n$ (with $n \ge k$).

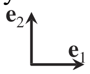
Remark 6. We recall that *spanS* means the smallest linear subspace that contains the set $S \subseteq \mathbb{E}_n$.

Thus, a 2×2 determinant can be considered as the Clifford ratio between the two coordinate-free blades $\mathbf{u} \wedge \mathbf{v}$ and I_2 (a "blade" being the geometric product of nonzero mutually orthogonal vectors). Those elements are also called "$\mathbb{G}_2$-pseudo-scalars," because they are scalar multiples of the orientation I_2 of $\mathbb{E}_2$ (the Euclidean space generating $\mathbb{G}_2$) and can be interpreted as oriented areas in $span\{\mathbf{e}_1, \mathbf{e}_2\} = \mathbb{E}_2$. Later, we will see that this Clifford geometric interpretation of a 2×2 determinant holds for every $k \times k$ determinant. Let us also observe that if $\mathbf{v} = \nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2$, then

$$\begin{aligned}\mathbf{v}I_2 &= (\nu_1\mathbf{e}_1 + \nu_2\mathbf{e}_2)\mathbf{e}_1\mathbf{e}_2 = \nu_1\mathbf{e}_1\mathbf{e}_1\mathbf{e}_2 + \nu_2\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2 = -\nu_1\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1 - \nu_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_2 = -\mathbf{e}_1\mathbf{e}_2(\nu_1\mathbf{e}_1 + \mathbf{e}_2) = -I_2\mathbf{v} \\ &= -\nu_2\mathbf{e}_1 + \nu_1\mathbf{e}_2 \in span\{\mathbf{e}_1, \mathbf{e}_2\}\end{aligned}$$

Thus, we can also write a 2×2 determinant as a scalar product

$$\det\begin{pmatrix} \mu_1 & \mu_2 \\ \nu_1 & \nu_2 \end{pmatrix} = (\mathbf{u} \wedge \mathbf{v})(I_2)^{-1} = -\frac{1}{2}(\mathbf{uv} - \mathbf{vu})I_2 = \frac{1}{2}(-\mathbf{uv}I_2 + \mathbf{vu}I_2) = \frac{1}{2}(\mathbf{u}I_2\mathbf{v} + \mathbf{vu}I_2) = (\mathbf{u}I_2) \cdot \mathbf{v} = -\left[\mathbf{u}(I_2)^{-1}\right] \cdot \mathbf{v}$$

where $\mathbf{u}I_2$ is the vector obtained by rotating vector $\mathbf{u}$ of a right angle counterclockwise in $span\{\mathbf{e}_1, \mathbf{e}_2\}$ (provided $\mathbf{e}_1$ and $\mathbf{e}_2$ are mutually oriented like that: ).

3 | THE CASE OF A TWO-VARIABLE FUNCTION

The possibility to express the determinant of a square matrix both as a Clifford quotient and as a scalar product is the key tool to write in a coordinate-free framework the equation of a plane secant the graph of a multivariable function.

3.1 | Coordinate-free expression of a plane secant the graph of a two-variable function

Let $f : \Omega \subseteq \mathbb{E}_2 \to \mathbb{R}$ be a function defined on a subset Ω of the 2-dimensional Euclidean space $\mathbb{E}_2$. A plane passing through the three points of $\mathbb{E}_2 \oplus \mathbb{R} \simeq \mathbb{R}^3$

$$(\mathbf{a}, f(\mathbf{a})) = (\alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2, f(\mathbf{a})) \ , \ (\mathbf{b}, f(\mathbf{b})) = (\beta_1\mathbf{e}_1 + \beta_2\mathbf{e}_2, f(\mathbf{b})) \ , \ (\mathbf{c}, f(\mathbf{c})) = (\gamma_1\mathbf{e}_1 + \gamma_2\mathbf{e}_2, f(\mathbf{c}))$$

can be represented by the Cartesian relation

$$\det\begin{pmatrix} x - \alpha_1 & y - \alpha_2 & z - f(\mathbf{a}) \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 & f(\mathbf{b}) - f(\mathbf{a}) \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 & f(\mathbf{c}) - f(\mathbf{a}) \end{pmatrix} = 0 \tag{3.1}$$

between the real variables $x, y, z \in \mathbb{R}$. This determinant can be rewritten by a Laplace expansion as follows:

$$[z - f(\mathbf{a})]\det\begin{pmatrix} \beta_1 - \alpha_1 & \beta_2 - \alpha_2 \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 \end{pmatrix} - [f(\mathbf{b}) - f(\mathbf{a})]\det\begin{pmatrix} x - \alpha_1 & y - \alpha_2 \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 \end{pmatrix} + [f(\mathbf{c}) - f(\mathbf{a})]\det\begin{pmatrix} x - \alpha_1 & y - \alpha_2 \\ \beta_1 - \alpha_1 & \beta_2 - \alpha_2 \end{pmatrix}$$

Then, in $\mathbb{G}_2$, Equation (3.1) becomes

$$[z - f(\mathbf{a})]\,[(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})]\,(I_2)^{-1} - [f(\mathbf{b}) - f(\mathbf{a})]\,[(\mathbf{x} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})]\,(I_2)^{-1} + [f(\mathbf{c}) - f(\mathbf{a})]\,[(\mathbf{x} - \mathbf{a}) \wedge (\mathbf{b} - \mathbf{a})]\,(I_2)^{-1} = 0$$

being $\mathbf{x} = x\mathbf{e}_1 + y\mathbf{e}_2 \in \mathbb{E}_2$, and $(\mathbf{x}, z) \in \mathbb{E}_2 \oplus \mathbb{R}$. The foregoing relation is equivalent, in $\mathbb{G}_2$, to

$$[z - f(\mathbf{a})]\,[(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a})] = [f(\mathbf{c}) - f(\mathbf{a})]\,[(\mathbf{b} - \mathbf{a}) \wedge (\mathbf{x} - \mathbf{a})] - [f(\mathbf{b}) - f(\mathbf{a})]\,[(\mathbf{c} - \mathbf{a}) \wedge (\mathbf{x} - \mathbf{a})]$$

Let us define

$$\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = (\mathbf{b} - \mathbf{a}) \wedge (\mathbf{c} - \mathbf{a}) \ \ \left[\text{which is also equal to } (\mathbf{a} - \mathbf{b}) \wedge (\mathbf{b} - \mathbf{c})\right]$$

We observe that

$$\tau_2 = \frac{1}{2}\det\begin{pmatrix} \beta_1 - \alpha_1 & \beta_2 - \alpha_2 \\ \gamma_1 - \alpha_1 & \gamma_2 - \alpha_2 \end{pmatrix} = \frac{1}{2}\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}(I_2)^{-1}$$

is the oriented area of the triangle having vertices $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ (the subscript "2" in τ_2 anticipate the use of τ_n as the hypervolume of a n-dimensional simplex). So, we can write $\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = 2\tau_2 I_2$, and $\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = \frac{1}{2\tau_2}(I_2)^{-1}$. Then, the equation of the secant plane (3.1) becomes

$$2\tau_2\,[z - f(\mathbf{a})]\,I_2 = \{[f(\mathbf{c}) - f(\mathbf{a})]\,(\mathbf{b} - \mathbf{a}) - [f(\mathbf{b}) - f(\mathbf{a})]\,(\mathbf{c} - \mathbf{a})\} \wedge (\mathbf{x} - \mathbf{a})$$

That is,

$$\begin{aligned} z &= f(\mathbf{a}) + \frac{1}{2\tau_2}\{\{[f(\mathbf{c}) - f(\mathbf{a})](\mathbf{b}-\mathbf{a}) - [f(\mathbf{b}) - f(\mathbf{a})](\mathbf{c}-\mathbf{a})\} \wedge (\mathbf{x}-\mathbf{a})\}(I_2)^{-1} \\ &= f(\mathbf{a}) + \frac{1}{2\tau_2}\left\{\{[f(\mathbf{b}) - f(\mathbf{a})](\mathbf{c}-\mathbf{a}) - [f(\mathbf{c}) - f(\mathbf{a})](\mathbf{b}-\mathbf{a})\}(I_2)^{-1}\right\} \cdot (\mathbf{x}-\mathbf{a}) \\ &= f(\mathbf{a}) + \left\{\{[f(\mathbf{b}) - f(\mathbf{a})](\mathbf{c}-\mathbf{a}) - [f(\mathbf{c}) - f(\mathbf{a})](\mathbf{b}-\mathbf{a})\}\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}\right\} \cdot (\mathbf{x}-\mathbf{a}) \end{aligned}$$

Thus, we have that

$$\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = [[f(\mathbf{b}) - f(\mathbf{a})](\mathbf{c}-\mathbf{a}) - [f(\mathbf{c}) - f(\mathbf{a})](\mathbf{b}-\mathbf{a})]\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}$$

which is, in fact, the Clifford ratio between the multidifference vector

$$\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = [f(\mathbf{b}) - f(\mathbf{a})](\mathbf{c}-\mathbf{a}) - [f(\mathbf{c}) - f(\mathbf{a})](\mathbf{b}-\mathbf{a}) \in \mathbb{E}_2$$

and the bivector

$$\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = (\mathbf{b}-\mathbf{a}) \wedge (\mathbf{c}-\mathbf{a}) = (\mathbf{a}-\mathbf{b}) \wedge (\mathbf{b}-\mathbf{c}) = 2\tau_2 I_2$$

Remark 7. As $\mathbf{r}_{(f,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)} = \left(\Delta f_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}\right)\left(\Delta_{(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}\right)^{-1}$, you can verify that

$$\mathbf{r}_{\left(f,\mathbf{v}_{\sigma_1},\mathbf{v}_{\sigma_2},\mathbf{v}_{\sigma_3}\right)} = \mathbf{r}_{(f,\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3)}$$

for every triple of non-collinear vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \Omega$, and every permutation $\sigma \in \mathcal{S}_3$ of the set $\{1, 2, 3\}$. In other words, vector $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ is a totally symmetric function of its vector arguments $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{E}_2$, as the scalar difference quotient $\frac{\Delta f_{(a,b)}}{\Delta_{(a,b)}} = \frac{f(a)-f(b)}{a-b}$ is a totally symmetric function of its scalar arguments $a, b \in \mathbb{R}$.

Remark 8. In the case of Example 1, we have that

- $\mathbf{b}-\mathbf{a} = -\alpha\mathbf{e}_1 + \beta\mathbf{e}_2$, $\mathbf{c}-\mathbf{a} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2$
- $f(\mathbf{b}) - f(\mathbf{a}) = \sqrt{1-\alpha^2} - 1 = f(\mathbf{c}) - f(\mathbf{a})$
- $\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = 2\alpha\left(\sqrt{1-\alpha^2} - 1\right)\mathbf{e}_1$
- $\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = -2\alpha\beta\mathbf{e}_1\mathbf{e}_2$

So,

$$\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{\sqrt{1-\alpha^2} - 1}{\beta}\mathbf{e}_2$$

3.2 | The vector $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}$ as linear combination of vectors a, b, and c

Given an oriented triangle in $\mathbb{E}_2$ whose vertices are the ordered vectors **a**, **b**, and **c** in $\mathbb{E}_2$, we define

$$\Delta\mathbf{b} = \mathbf{b}-\mathbf{a},\ \Delta\mathbf{c} = \mathbf{c}-\mathbf{a},\ \Delta f_{(\mathbf{b})} = f(\mathbf{b}) - f(\mathbf{a}), \text{ and } \Delta f_{(\mathbf{c})} = f(\mathbf{c}) - f(\mathbf{a})$$

Let us recall that

- $\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = (\Delta\mathbf{b}) \wedge (\Delta\mathbf{c}) = 2\tau_2 I_2$
- $\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^2 = -4\tau_2^2 = |\Delta\mathbf{b}|^2|\Delta\mathbf{c}|^2 - (\Delta\mathbf{b}\cdot\Delta\mathbf{c})^2$
- $\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = \frac{1}{\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^2}\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = -\frac{1}{4\tau_2^2}\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = -\frac{1}{2\tau_2}I_2$
- $\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = \Delta f_{(\mathbf{b})}\Delta\mathbf{c} - \Delta f_{(\mathbf{c})}\Delta\mathbf{b}$
- $\mathbf{u}(\mathbf{v}\wedge\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{w} - (\mathbf{u}\cdot\mathbf{w})\mathbf{v}$ for all **u**, **v**, and **w** in $\mathbb{E}_2$.

So, we can write

$$\begin{aligned}
\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} &= \left(\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = \left(\Delta f_{(\mathbf{b})}\Delta\mathbf{c} - \Delta f_{(\mathbf{c})}\Delta\mathbf{b}\right)[(\Delta\mathbf{b})\wedge(\Delta\mathbf{c})]^{-1} \\
&= -\frac{1}{4\tau_2^2}\left(\Delta f_{(\mathbf{b})}\Delta\mathbf{c} - \Delta f_{(\mathbf{c})}\Delta\mathbf{b}\right)[(\Delta\mathbf{b})\wedge(\Delta\mathbf{c})] \\
&= -\frac{1}{4\tau_2^2}\left\{\Delta f_{(\mathbf{b})}\Delta\mathbf{c}\,[(\Delta\mathbf{b})\wedge(\Delta\mathbf{c})] - \Delta f_{(\mathbf{c})}\Delta\mathbf{b}\,[(\Delta\mathbf{b})\wedge(\Delta\mathbf{c})]\right\} \\
&= -\frac{1}{4\tau_2^2}\left\{\Delta f_{(\mathbf{b})}(\Delta\mathbf{c}\cdot\Delta\mathbf{b})\Delta\mathbf{c}) - \Delta f_{(\mathbf{b})}|\Delta\mathbf{c}|^2\Delta\mathbf{b}\right\} + \frac{1}{4\tau_2^2}\left\{\Delta f_{(\mathbf{c})}|\Delta\mathbf{b}|^2\Delta\mathbf{c} - \Delta f_{(\mathbf{c})}(\Delta\mathbf{b}\cdot\Delta\mathbf{c})\Delta\mathbf{b})\right] \\
&= \frac{\Delta f_{(\mathbf{b})}|\Delta\mathbf{c}|^2 - \Delta f_{(\mathbf{c})}(\Delta\mathbf{b}\cdot\Delta\mathbf{c})}{|\Delta\mathbf{b}|^2|\Delta\mathbf{c}|^2 - (\Delta\mathbf{b}\cdot\Delta\mathbf{c})^2}\Delta\mathbf{b} + \frac{\Delta f_{(\mathbf{c})}|\Delta\mathbf{b}|^2 - \Delta f_{(\mathbf{b})}(\Delta\mathbf{b}\cdot\Delta\mathbf{c})}{|\Delta\mathbf{b}|^2|\Delta\mathbf{c}|^2 - (\Delta\mathbf{b}\cdot\Delta\mathbf{c})^2}\Delta\mathbf{c}
\end{aligned}$$

3.3 | Mirroring vectors and points I

3.3.1 | Vector mirrored by a 1-dimensional linear subspace

We recall that to each nonzero vector $\mathbf{u} \in \mathbb{E}_n$ we can associate the 1-dimensional linear subspace of $\mathbb{E}_n$ $span\{\mathbf{u}\} = \{\lambda\mathbf{u} : \lambda \in \mathbb{R}\} = \mathbb{R}\mathbf{u}$. Besides, every nonzero vector $\mathbf{u} \in \mathbb{E}_n$ is invertible in $\mathbb{G}_n$, and $\mathbf{u}^{-1} = \frac{1}{|\mathbf{u}|^2}\mathbf{u}$. Then, given a vector $\mathbf{v} \in \mathbb{E}_n$, we can write

$$\mathbf{v} = \mathbf{v}\mathbf{u}\mathbf{u}^{-1} = (\mathbf{v}\mathbf{u})\mathbf{u}^{-1} = (\mathbf{v}\cdot\mathbf{u} + \mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1} = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} + (\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1} \tag{3.2}$$

Remark 9. If $\mathbf{v} \in \mathbb{E}_n$ is invertible and $\alpha \in \mathbb{R}$ the expression

$$\frac{\alpha}{\mathbf{v}}$$

is unambiguous, because in $\mathbb{G}_n$ scalars commute with vectors (and with any other element, indeed). As a matter of fact,

$$\frac{\alpha}{\mathbf{v}} = \alpha\mathbf{v}^{-1} = \mathbf{v}^{-1}\alpha = \frac{\alpha}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{\alpha}{|\mathbf{v}|^2}\mathbf{v}$$

Remark 10. Notice that $(\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1}$ is orthogonal to $\mathbf{u}$. As a matter of fact,

$$\begin{aligned}
4\left[(\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1}\right]\cdot\mathbf{u} &= 2(\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1}\mathbf{u} + 2\mathbf{u}(\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1} = 2(\mathbf{v}\wedge\mathbf{u}) + \mathbf{u}(\mathbf{v}\mathbf{u} - \mathbf{u}\mathbf{v})\mathbf{u}^{-1} = 2(\mathbf{v}\wedge\mathbf{u}) + \mathbf{u}\mathbf{v} - \mathbf{u}\mathbf{u}\mathbf{v}\mathbf{u}^{-1} \\
&= 2(\mathbf{v}\wedge\mathbf{u}) + \mathbf{u}\mathbf{v} - \mathbf{v}\mathbf{u} = 2(\mathbf{v}\wedge\mathbf{u}) + 2(\mathbf{u}\wedge\mathbf{v}) = 0
\end{aligned}$$

As $\frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u}$ is parallel to $\mathbf{u}$, and $(\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1}$ is orthogonal to $\mathbf{u}$, we can consider relation (3.2) as the decomposition of $\mathbf{v} = \mathbf{v}_{\parallel} + \mathbf{v}_{\perp}$ into its orthogonal projection $\mathbf{v}_{\parallel} = \frac{\mathbf{v}\cdot\mathbf{u}}{|\mathbf{u}|^2}\mathbf{u} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u}^{-1}$ parallel to the line $\mathbb{R}\mathbf{u} = \mathcal{L}_{(\mathbf{0},\mathbf{u})}$, and its rejection $\mathbf{v}_{\perp} = (\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1}$ orthogonal to $\mathbb{R}\mathbf{u}$ (as shown in Figure 1). Thus, the vector $\hat{\mathbf{v}}$, obtained by mirroring $\mathbf{v}$ through $\mathcal{L}_{(\mathbf{0},\mathbf{u})}$ (see Figure 2), can be written as

$$\begin{aligned}
\hat{\mathbf{v}} &= \mathbf{v}_{\parallel} - \mathbf{v}_{\perp} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u}^{-1} - (\mathbf{v}\wedge\mathbf{u})\mathbf{u}^{-1} = (\mathbf{u}\cdot\mathbf{v})\mathbf{u}^{-1} + (\mathbf{u}\wedge\mathbf{v})\mathbf{u}^{-1} = (\mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v})\mathbf{u}^{-1} = \mathbf{u}\mathbf{v}\mathbf{u}^{-1} \\
&= [2(\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\mathbf{u}]\,\mathbf{u}^{-1} = 2(\mathbf{u}\cdot\mathbf{v})\mathbf{u}^{-1} - \mathbf{v} = 2\frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}|^2}\mathbf{u} - \mathbf{v} \in span\{\mathbf{u},\mathbf{v}\}
\end{aligned}$$

as for each $\mathbf{u},\mathbf{v} \in \mathbb{E}_n$ $\mathbf{u}\mathbf{v} = 2(\mathbf{u}\cdot\mathbf{v}) - \mathbf{v}\mathbf{u}$. Moreover, $|\hat{\mathbf{v}}| = |\mathbf{v}|$; as a matter of fact,

$$|\hat{\mathbf{v}}|^2 = \hat{\mathbf{v}}\hat{\mathbf{v}} = \mathbf{u}\mathbf{v}\mathbf{u}^{-1}\mathbf{u}\mathbf{v}\mathbf{u}^{-1} = \mathbf{u}|\mathbf{v}|^2\mathbf{u}^{-1} = |\mathbf{v}|^2$$

3.3.2 | Point mirrored by a line in $\mathbb{E}_n$ (with $n \geq 2$)

Given three non-collinear points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ in $\mathbb{E}_n$, we want to mirror point $\mathbf{a}$ by the line $\mathcal{L}_{(\mathbf{b},\mathbf{c})}$ passing through the points $\mathbf{b}$ and $\mathbf{c}$. We denote by $\bar{\mathbf{a}}$ the mirrored point (see Figure 3). We can express $\bar{\mathbf{a}}$ by computing the vector $\hat{\mathbf{v}}$, obtained by

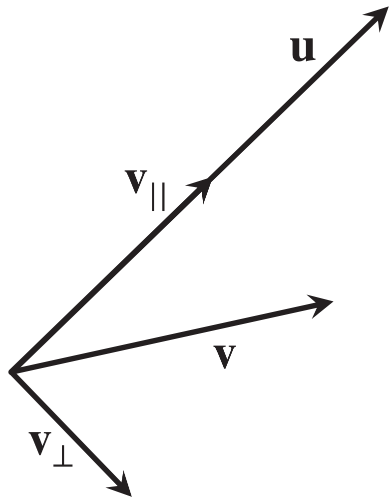


FIGURE 1 Deomposition of $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$; $\mathbf{v}_{\|}$ is parallel to $\mathbf{u}$, $\mathbf{v}_{\perp}$ is orthogonal to $\mathbf{u}$.

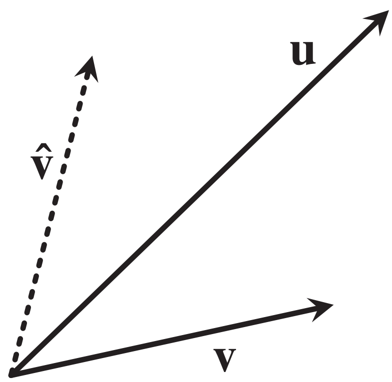


FIGURE 2 Mirroring vector $\mathbf{v}$ through line $\mathcal{L}_{(\mathbf{0},\mathbf{u})}$.

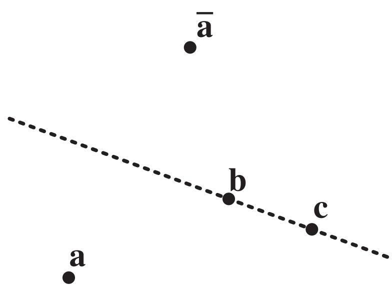


FIGURE 3 Point mirrored by a line.

mirroring vector $\mathbf{v} = \mathbf{a} - \mathbf{b}$ by the nonzero vector $\mathbf{c} - \mathbf{b}$. Thus, the reflected point $\bar{\mathbf{a}}$ can be expressed using the geometric Clifford product in $\mathbb{G}_n$

$$\begin{aligned}\bar{\mathbf{a}} &= \mathbf{b} + (\mathbf{c}-\mathbf{b})(\mathbf{a}-\mathbf{b})(\mathbf{c}-\mathbf{b})^{-1} = \mathbf{b} - 2\left[(\mathbf{c}-\mathbf{b})\cdot(\Delta\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1} + (\mathbf{b}-\mathbf{a}) \\ &= 2\mathbf{b} - 2\left[(\mathbf{c}-\mathbf{b})\cdot(\Delta\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1} - \mathbf{a} = 2\mathbf{b} - 2\frac{(\mathbf{c}-\mathbf{b})\cdot(\Delta\mathbf{b})}{\mathbf{c}-\mathbf{b}} - \mathbf{a} = 2\mathbf{b} - 2\frac{(\mathbf{c}-\mathbf{b})\cdot(\Delta\mathbf{b})}{|\mathbf{c}-\mathbf{b}|^2}(\mathbf{c}-\mathbf{b}) - \mathbf{a} \in \mathcal{L}_{(\mathbf{a},\mathbf{b},\mathbf{c})}\end{aligned}$$

Thus,

$$\begin{aligned}\bar{\mathbf{a}}-\mathbf{a} &= 2(\Delta\mathbf{b}) - 2\left[(\mathbf{c}-\mathbf{b})\cdot(\Delta\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1} = 2(\Delta\mathbf{b})(\mathbf{c}-\mathbf{b})(\mathbf{c}-\mathbf{b})^{-1} - 2\left[(\Delta\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1}\\ &= 2\left\{(\Delta\mathbf{b})(\mathbf{c}-\mathbf{b}) - \left[(\Delta\mathbf{b})\cdot(\mathbf{c}-\mathbf{b})\right]\right\}(\mathbf{c}-\mathbf{b})^{-1} = 2\left[(\Delta\mathbf{b})\wedge(\mathbf{c}-\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1}\end{aligned}$$

and $|\bar{\mathbf{a}}-\mathbf{b}| = |\mathbf{a}-\mathbf{b}|$.

Remark 11. By using Remark 10, you can verify that vectors $\bar{\mathbf{a}}-\mathbf{a}$ and $\mathbf{b}-\mathbf{c}$ are mutually orthogonal. Moreover,

$$(\bar{\mathbf{a}}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{b}) = (\bar{\mathbf{a}}-\mathbf{a})(\mathbf{c}-\mathbf{b}) = 2\left[(\Delta\mathbf{b})\wedge(\mathbf{c}-\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1}(\mathbf{c}-\mathbf{b}) = 2(\mathbf{b}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{b}) = 2\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}$$

3.4 | The mean multidifference vector

Let us define the "mean multidifference vector,"

$$\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{1}{2}\left(\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} + \Delta f_{(\bar{\mathbf{a}},\mathbf{c},\mathbf{b})}\right) = \frac{1}{2}\left(\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} - \Delta f_{(\bar{\mathbf{a}},\mathbf{b},\mathbf{c})}\right)$$

Lemma 1.

$$\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{1}{2}\left\{\left[f(\bar{\mathbf{a}}) - f(\mathbf{a})\right](\mathbf{c}-\mathbf{b}) - \left[f(\mathbf{c}) - f(\mathbf{b})\right](\bar{\mathbf{a}}-\mathbf{a})\right\}$$

Proof of Lemma 1.

$$\begin{aligned}\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} &= \frac{1}{2}\left\{\left[f(\mathbf{b}) - f(\mathbf{a})\right](\mathbf{c}-\mathbf{a}) - \left[f(\mathbf{c}) - f(\mathbf{a})\right](\mathbf{b}-\mathbf{a})\right\} - \frac{1}{2}\left\{\left[f(\mathbf{b}) - f(\bar{\mathbf{a}})\right](\mathbf{c}-\bar{\mathbf{a}}) - \left[f(\mathbf{c}) - f(\bar{\mathbf{a}})\right](\mathbf{b}-\bar{\mathbf{a}})\right\}\\ &= \frac{1}{2}\left\{\left[f(\bar{\mathbf{a}}) - f(\mathbf{a})\right]\mathbf{c} + \left[f(\mathbf{c}) - f(\mathbf{b})\right]\mathbf{a} + \left[f(\mathbf{a}) - f(\bar{\mathbf{a}})\right]\mathbf{b} + \left[f(\mathbf{b}) - f(\mathbf{c})\right]\bar{\mathbf{a}}\right\}\end{aligned}$$

The foregoing Lemma 1 and Remark 9 allow us to obtain a simple expression for the vector corresponding to the mean multidifference quotient $\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}$, which strongly recall the usual difference quotients (as anticipated in the abstract and in the introduction of this work).

Proposition 1.

$$\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{f(\bar{\mathbf{a}}) - f(\mathbf{a})}{\bar{\mathbf{a}}-\mathbf{a}} + \frac{f(\mathbf{c}) - f(\mathbf{b})}{\mathbf{c}-\mathbf{b}}$$

Proof of Proposition 1. From Remark 11, we have that $\bar{\mathbf{a}}-\mathbf{a}$ is orthogonal to $\mathbf{c}-\mathbf{b}$, and $(\bar{\mathbf{a}}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{b}) = 2\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}$. So, we can write

$$\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{1}{2}(\bar{\mathbf{a}}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{b}) = \frac{1}{2}(\bar{\mathbf{a}}-\mathbf{a})(\mathbf{c}-\mathbf{b}) \text{ and } \left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = 2(\mathbf{c}-\mathbf{b})^{-1}(\bar{\mathbf{a}}-\mathbf{a})^{-1} = -2(\bar{\mathbf{a}}-\mathbf{a})^{-1}(\mathbf{c}-\mathbf{b})^{-1}$$

Then, by Lemma 1, we can write

$$\begin{aligned}\bar{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} &= \left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = \left\{\left[f(\bar{\mathbf{a}}) - f(\mathbf{a})\right](\mathbf{c}-\mathbf{b}) - \left[f(\mathbf{c}) - f(\mathbf{b})\right](\bar{\mathbf{a}}-\mathbf{a})\right\}(\mathbf{c}-\mathbf{b})^{-1}(\bar{\mathbf{a}}-\mathbf{a})^{-1}\\ &= \left[f(\bar{\mathbf{a}}) - f(\mathbf{a})\right](\bar{\mathbf{a}}-\mathbf{a})^{-1} + \left[f(\mathbf{c}) - f(\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1} = \frac{f(\bar{\mathbf{a}}) - f(\mathbf{a})}{\bar{\mathbf{a}}-\mathbf{a}} + \frac{f(\mathbf{c}) - f(\mathbf{b})}{\mathbf{c}-\mathbf{b}}\end{aligned}$$

coherently with Remark 9. □

3.5 | Convergence of the mean secant plane to the tangent plane

As we have seen, in the example, the local Schwarz paradox is due to the nonexistence of the limit

$$\lim_{\substack{(\mathbf{a},\mathbf{b},\mathbf{c})\to(\mathbf{x}_0,\mathbf{x}_0,\mathbf{x}_0)\\ \mathbf{a},\mathbf{b},\mathbf{c}\ \text{not collinear}}} \left(\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}$$

Here, we recall[22] the definition of strong (or strict) differentiability of a multivariable function at an internal point of its domain.

Definition 1. A function $f : \Omega \subseteq \mathbb{E}_n \to \mathbb{R}$ is strongly differentiable at $\mathbf{x}_0$ (a point internal to Ω) if there exists a vector $\mathbf{f}^*(\mathbf{x}_0) \in \mathbb{E}_n$ such that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|\mathbf{u}-\mathbf{x}_0| < \delta$ and $|\mathbf{v}-\mathbf{x}_0| < \delta$, then

$$\left|f(\mathbf{u}) - f(\mathbf{v}) - \mathbf{f}^*(\mathbf{x}_0)\cdot(\mathbf{u}-\mathbf{v})\right| \le \epsilon\,|\mathbf{u}-\mathbf{v}|$$

being $\mathbf{u},\mathbf{v} \in \Omega$.

Remark 12. We recall that, if a function is strongly differentiable at $\mathbf{x}_0$, then it is also differentiable, and the vector $\mathbf{f}^*(\mathbf{x}_0)$ coincides with the gradient $\nabla f(\mathbf{x}_0)$.

Theorem 1. *If the function $f : \Omega \subseteq \mathbb{E}_2 \to \mathbb{R}$ is strongly differentiable at $\mathbf{x}_0$ (a point internal to Ω), then*

$$\lim_{\substack{(\mathbf{a},\mathbf{b},\mathbf{c})\to(\mathbf{x}_0,\mathbf{x}_0,\mathbf{x}_0)\\ \mathbf{a},\mathbf{b},\mathbf{c}\ \textit{not collinear}}} \left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = \nabla f(\mathbf{x}_0)$$

Remark 13. The foregoing result states that the "mean secant plane"

$$z = f(\mathbf{a}) + \overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}\cdot(\mathbf{x}-\mathbf{a})$$

where $\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \left(\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}$, always converges to the tangent plane

$$z = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$$

as the nondegenerate triangle with vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ contracts to point $\mathbf{x}_0$.

Proof of Theorem 1. Let us recall that, given three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $\mathbb{E}_n$, if $\mathbf{u} \in span\{\mathbf{v},\mathbf{w}\}$, then $\mathbf{u}(\mathbf{v}\wedge\mathbf{w}) = (\mathbf{u}\cdot\mathbf{v})\mathbf{w} - (\mathbf{u}\cdot\mathbf{w})\mathbf{v}$. So, we can write

$$\nabla f(\mathbf{x}_0)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right) = \frac{1}{2}\nabla f(\mathbf{x}_0)\left[(\bar{\mathbf{a}}-\mathbf{a})\wedge(\mathbf{c}-\mathbf{b})\right] = \frac{1}{2}\left[\nabla f(\mathbf{x}_0)\cdot(\bar{\mathbf{a}}-\mathbf{a})\right](\mathbf{c}-\mathbf{b}) - \frac{1}{2}\left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{c}-\mathbf{b})\right](\bar{\mathbf{a}}-\mathbf{a})$$

As $\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = 2(\mathbf{c}-\mathbf{b})^{-1}(\bar{\mathbf{a}}-\mathbf{a})^{-1} = -2(\bar{\mathbf{a}}-\mathbf{a})^{-1}(\mathbf{c}-\mathbf{b})^{-1}$, we can write

$$\begin{aligned}\nabla f(\mathbf{x}_0) &= \nabla f(\mathbf{x}_0)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1} = \left[\nabla f(\mathbf{x}_0)\cdot(\bar{\mathbf{a}}-\mathbf{a})\right](\bar{\mathbf{a}}-\mathbf{a})^{-1} + \left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{c}-\mathbf{b})\right](\mathbf{c}-\mathbf{b})^{-1}\\ &= \frac{\nabla f(\mathbf{x}_0)\cdot(\bar{\mathbf{a}}-\mathbf{a})}{\bar{\mathbf{a}}-\mathbf{a}} + \frac{\nabla f(\mathbf{x}_0)\cdot(\mathbf{c}-\mathbf{b})}{\mathbf{c}-\mathbf{b}}\end{aligned}$$

[22] See, for example, [14–16].

So, by Proposition 1, we can write

$$\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} - \nabla f(\mathbf{x}_0) = [f(\bar{\mathbf{a}}) - f(\mathbf{a}) - \nabla f(\mathbf{x}_0)\cdot(\bar{\mathbf{a}}-\mathbf{a})]\,(\bar{\mathbf{a}}-\mathbf{a})^{-1} + [f(\mathbf{c}) - f(\mathbf{b}) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{c}-\mathbf{b})]\,(\mathbf{c}-\mathbf{b})^{-1}$$
$$= \frac{f(\bar{\mathbf{a}}) - f(\mathbf{a}) - \nabla f(\mathbf{x}_0)\cdot(\bar{\mathbf{a}}-\mathbf{a})}{\bar{\mathbf{a}}-\mathbf{a}} + \frac{f(\mathbf{c}) - f(\mathbf{b}) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{c}-\mathbf{b})}{\mathbf{c}-\mathbf{b}}$$

As $\left|\frac{\alpha}{\mathbf{v}}\right| = \frac{|\alpha|}{|\mathbf{v}|}$ for every $\alpha \in \mathbb{R}$ and for every invertible vector $\mathbf{v} \in \mathbb{E}_n$, we have that

$$\left|\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} - \nabla f(\mathbf{x}_0)\right| \leq \frac{|f(\bar{\mathbf{a}}) - f(\mathbf{a}) - \nabla f(\mathbf{x}_0)\cdot(\bar{\mathbf{a}}-\mathbf{a})|}{|\bar{\mathbf{a}}-\mathbf{a}|} + \frac{|f(\mathbf{c}) - f(\mathbf{b}) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{c}-\mathbf{b})|}{|\mathbf{c}-\mathbf{b}|}$$

As f is strongly differentiable at $\mathbf{x}_0$, we know that, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that if $|\mathbf{u}-\mathbf{x}_0| < \delta_\epsilon$ and $|\mathbf{v}-\mathbf{x}_0| < \delta_\epsilon$, then

$$|f(\mathbf{u}) - f(\mathbf{v}) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{u}-\mathbf{v})| \leq \epsilon|\mathbf{u}-\mathbf{v}|$$

So, if we choose $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ non-collinear, and such that

- $|\mathbf{a}-\mathbf{x}_0| < \frac{1}{3}\left(\delta_{\frac{\epsilon}{2}}\right)$
- $|\mathbf{b}-\mathbf{x}_0| < \frac{1}{3}\left(\delta_{\frac{\epsilon}{2}}\right)$
- $|\mathbf{c}-\mathbf{x}_0| < \delta_{\frac{\epsilon}{2}}$,

then

$$\left|\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} - \nabla f(\mathbf{x}_0)\right| \leq \epsilon$$

as

- $|\mathbf{a}-\mathbf{b}| \leq |\mathbf{a}-\mathbf{x}_0| + |\mathbf{b}-\mathbf{x}_0| < \frac{2}{3}\left(\delta_{\frac{\epsilon}{2}}\right)$
- $|\bar{\mathbf{a}}-\mathbf{x}_0| \leq |\bar{\mathbf{a}}-\mathbf{b}| + |\mathbf{b}-\mathbf{x}_0| = |\mathbf{a}-\mathbf{b}| + |\mathbf{b}-\mathbf{x}_0| < \delta_{\frac{\epsilon}{2}}$

□

Remark 14. The divergence phenomenon of the local Schwarz paradox is due to the fact that vectors $\mathbf{b}-\mathbf{a}$ and $\mathbf{c}-\mathbf{a}$ are not mutually orthogonal, in general. On the contrary, vectors $\mathbf{a}-\bar{\mathbf{a}}$ and $\mathbf{b}-\mathbf{c}$ are always orthogonal. Moreover, the following crucial identities hold:

$$(\bar{\mathbf{a}}-\mathbf{a})(\mathbf{c}-\mathbf{b}) = (\bar{\mathbf{a}}-\mathbf{a}) \wedge (\mathbf{c}-\mathbf{b}) = 2(\mathbf{b}-\mathbf{a}) \wedge (\mathbf{c}-\mathbf{a})$$

Such key properties of the mirrored points will be used to extend the foregoing convergence result to higher dimensions.

Remark 15. In the case of Example 1, we have that

- $\bar{\mathbf{a}} = 2\beta\mathbf{e}_2\ ,\ f(\bar{\mathbf{a}}) = 1$
- $\overline{\Delta} f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = \mathbf{0}$

So, $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \mathbf{0}$, and $\nabla f(\mathbf{a}) = \mathbf{0}$, indeed.

In the same example, we could also choose the same points but in a different order. For example, one could choose

$$\mathbf{a} = -\alpha\mathbf{e}_1 + \beta\mathbf{e}_2, \mathbf{b} = \alpha\mathbf{e}_1 + \beta\mathbf{e}_2, \mathbf{c} = \mathbf{0}$$

In this case, the mirrored point would be different, and we would have

- $\Delta\mathbf{b} = \mathbf{b}-\mathbf{a} = 2\alpha\mathbf{e}_1\ ,\ \Delta\mathbf{c} = \mathbf{c}-\mathbf{a} = \alpha\mathbf{e}_1 - \beta\mathbf{e}_2\ ,\ \mathbf{c}-\mathbf{b} = -\mathbf{b}$
- $\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})} = (\mathbf{b}-\mathbf{a}) \wedge (\mathbf{c}-\mathbf{a}) = -2\alpha\beta\mathbf{e}_1\mathbf{e}_2$
- $f(\mathbf{b}) - f(\mathbf{a}) = 0\ ,\ f(\mathbf{c}) - f(\mathbf{a}) = 1 - \sqrt{1-\alpha^2}$

- $\Delta f_{(\mathbf{a},\mathbf{b},\mathbf{c})} = [f(\mathbf{b}) - f(\mathbf{a})](\mathbf{c}-\mathbf{a}) - [f(\mathbf{c}) - f(\mathbf{a})](\mathbf{b}-\mathbf{a}) = 2\alpha\left(\sqrt{1-\alpha^2}-1\right)\mathbf{e}_1$
- $\mathbf{r}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})} = \frac{1-\sqrt{1-\alpha^2}}{2\beta}\mathbf{e}_2$

as one would expect by Remark 7 and Remark 8; besides,

- $\bar{\mathbf{a}} = 2\mathbf{b} - 2[(\mathbf{c}-\mathbf{b})\cdot(\Delta\mathbf{b})](\mathbf{c}-\mathbf{b})^{-1} - \mathbf{a} = \frac{3\beta^2-\alpha^2}{\alpha^2+\beta^2}\alpha\mathbf{e}_1 + \frac{\beta^2-3\alpha^2}{\alpha^2+\beta^2}\beta\mathbf{e}_2$
- $\bar{\mathbf{a}} - \mathbf{a} = \frac{4\alpha\beta^2}{\alpha^2+\beta^2}\mathbf{e}_1 - \frac{4\alpha^2\beta}{\alpha^2+\beta^2}\mathbf{e}_2 \quad , \quad |\bar{\mathbf{a}}-\mathbf{a}|^2 = \frac{16\alpha^2\beta^2}{\alpha^2+\beta^2}$
- $f(\bar{\mathbf{a}}) = \sqrt{1-\alpha^2\left(\frac{3\beta^2-\alpha^2}{\alpha^2+\beta^2}\right)^2}$
-

$$\bar{\mathbf{r}}_{f_{(\mathbf{a},\mathbf{b},\mathbf{c})}} = \frac{f(\bar{\mathbf{a}})-f(\mathbf{a})}{|\bar{\mathbf{a}}-\mathbf{a}|^2}(\bar{\mathbf{a}}-\mathbf{a}) + \frac{f(\mathbf{c})-f(\mathbf{b})}{|\mathbf{c}-\mathbf{b}|^2}(\mathbf{c}-\mathbf{b}) =$$
$$= \left\{\alpha\beta^2\frac{\alpha^2-\beta^2}{\left(\alpha^2+\beta^2\right)^2}\frac{2}{\sqrt{1-\alpha^2\left(\frac{3\beta^2-\alpha^2}{\alpha^2+\beta^2}\right)^2}+\sqrt{1-\alpha^2}} - \frac{\alpha^3}{\left(\alpha^2+\beta^2\right)\left[1+\sqrt{1-\alpha^2}\right]}\right\}\mathbf{e}_1 +$$
$$-\left\{\alpha^2\beta\frac{\alpha^2-\beta^2}{\left(\alpha^2+\beta^2\right)^2}\frac{2}{\sqrt{1-\alpha^2\left(\frac{3\beta^2-\alpha^2}{\alpha^2+\beta^2}\right)^2}+\sqrt{1-\alpha^2}} + \frac{\alpha^2\beta}{\left(\alpha^2+\beta^2\right)\left[1+\sqrt{1-\alpha^2}\right]}\right\}\mathbf{e}_2 = \bar{\mathbf{r}}_{(\alpha,\beta)}$$

and you can verify that

$$\lim_{(\alpha,\beta)\to(0,0)} \bar{\mathbf{r}}_{(\alpha,\beta)} = \mathbf{0} = \nabla f(\mathbf{0})$$

4 | THE CASE OF A THREE-VARIABLE FUNCTION

4.1 | The 3×3 determinant as a Clifford quotient and as a scalar product

Let us now recall that the determinant of a 3×3 real matrix

$$\begin{pmatrix} \mu_{1,1} & \mu_{1,2} & \mu_{1,3} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} \end{pmatrix}$$

can be written as a coordinate-free Clifford quotient. As usual, let us fix any ordered triple $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of mutually orthonormal vectors in $\mathbb{E}_n$ (with $n \geq 3$), and let us consider $\mathbf{u}_1 = \mu_{1,1}\mathbf{e}_1 + \mu_{1,2}\mathbf{e}_2 + \mu_{1,3}\mathbf{e}_3$, $\mathbf{u}_2 = \mu_{2,1}\mathbf{e}_1 + \mu_{2,2}\mathbf{e}_2 + \mu_{2,3}\mathbf{e}_3$, and $\mathbf{u}_3 = \mu_{3,1}\mathbf{e}_1 + \mu_{3,2}\mathbf{e}_2 + \mu_{3,3}\mathbf{e}_3$, then you can verify that

$$(\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3)(I_3)^{-1} = \det\begin{pmatrix} \mu_{1,1} & \mu_{1,2} & \mu_{1,3} \\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3} \\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3} \end{pmatrix}$$

because

$$\mathbf{u}_{\sigma_1} \wedge \mathbf{u}_{\sigma_2} \wedge \mathbf{u}_{\sigma_3} = \epsilon_\sigma \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$$

for each permutation $\sigma \in \mathcal{S}_3$ of the set $\{1,2,3\}$, having parity $\epsilon_\sigma \in \{-1,1\}$, where

$$\begin{aligned}\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3 &= \frac{1}{6}(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 - \mathbf{u}_1\mathbf{u}_3\mathbf{u}_2 + \mathbf{u}_3\mathbf{u}_1\mathbf{u}_2 - \mathbf{u}_3\mathbf{u}_2\mathbf{u}_1 + \mathbf{u}_2\mathbf{u}_3\mathbf{u}_1 - \mathbf{u}_2\mathbf{u}_1\mathbf{u}_3)\\ &= \frac{1}{2}[(\mathbf{u}_1 \wedge \mathbf{u}_2)\mathbf{u}_3 + \mathbf{u}_3(\mathbf{u}_1 \wedge \mathbf{u}_2)] = \frac{1}{2}(\mathbf{u}_1\mathbf{u}_2\mathbf{u}_3 - \mathbf{u}_3\mathbf{u}_2\mathbf{u}_1)\\ \text{and } I_3 &= \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3, \text{ so that } (I_3)^{-1} = \mathbf{e}_3\mathbf{e}_2\mathbf{e}_1 = -I_3\end{aligned}$$

Remark 16. As I_2 before, also I_3 does not depend on the particular orthonormal basis chosen to define it in $span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \subseteq \mathbb{E}_n$, but only on the orientation of that basis. More precisely, if $\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}$ is any other orthonormal basis of $span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, then $\mathbf{g}_1 \wedge \mathbf{g}_2 \wedge \mathbf{g}_3 = \mathbf{g}_1\mathbf{g}_2\mathbf{g}_3$ is equal to I_3 or $-I_3$. That is why I_3 is called an “orientation” of $span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} = \mathbb{E}_3$.

Thus, a 3×3 determinant can be considered as the Clifford ratio between the two “3-blades” $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3$ and I_3 (a “k-blade” being the geometric product of k nonzero and mutually orthogonal vectors). Those elements can also be called “$\mathbb{G}_3$-pseudo-scalars,” as $\mathbb{G}_3$ is generated by $\mathbb{E}_3 = span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and can be interpreted as oriented volumes in $\mathbb{E}_3$. Let us observe that if

$$V = v_{1,2}\mathbf{e}_1\mathbf{e}_2 + v_{1,3}\mathbf{e}_1\mathbf{e}_3 + v_{2,3}\mathbf{e}_2\mathbf{e}_3 \in span\{\mathbf{e}_1\mathbf{e}_2, \mathbf{e}_1\mathbf{e}_3, \mathbf{e}_2\mathbf{e}_3\} = \mathbb{G}_{\binom{3}{2}}$$

(such elements in $\mathbb{G}_3$ are also called the “2-vectors”), then

$$\begin{aligned}VI_3 &= (v_{1,2}\mathbf{e}_1\mathbf{e}_2 + v_{1,3}\mathbf{e}_1\mathbf{e}_3 + v_{2,3}\mathbf{e}_2\mathbf{e}_3)\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = v_{1,2}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + v_{1,3}\mathbf{e}_1\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 + v_{2,3}\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\\ &= v_{1,2}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_2 + v_{1,3}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_1\mathbf{e}_3 + v_{2,3}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_2\mathbf{e}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3(v_{1,2}\mathbf{e}_1\mathbf{e}_2 + v_{1,3}\mathbf{e}_1\mathbf{e}_3 + v_{2,3}\mathbf{e}_2\mathbf{e}_3) = I_3V\\ &= -v_{1,2}\mathbf{e}_3 + v_{1,3}\mathbf{e}_2 - v_{2,3}\mathbf{e}_1 \in \mathbb{E}_3\end{aligned}$$

that is why the elements in $\mathbb{G}_{\binom{3}{2}}$ are also called “$\mathbb{G}_3$-pseudo-vectors”: Geometric multiplication by I_3 establishes a duality between vectors of $\mathbb{E}_3$ and elements of $\mathbb{G}_{\binom{3}{2}}$. In a similar way, you can verify that $\mathbf{u}I_3 = I_3\mathbf{u}$ for all $\mathbf{u} \in \mathbb{E}_3$. The foregoing properties allow us to write a 3×3 determinant as a scalar product

$$\begin{aligned}\det\begin{pmatrix}\mu_{1,1} & \mu_{1,2} & \mu_{1,3}\\ \mu_{2,1} & \mu_{2,2} & \mu_{2,3}\\ \mu_{3,1} & \mu_{3,2} & \mu_{3,3}\end{pmatrix} &= (\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3)(I_3)^{-1} = -\frac{1}{2}[(\mathbf{u}_1 \wedge \mathbf{u}_2)\mathbf{u}_3 + \mathbf{u}_3(\mathbf{u}_1 \wedge \mathbf{u}_2)]I_3\\ &= -\frac{1}{2}[(\mathbf{u}_1 \wedge \mathbf{u}_2)I_3\mathbf{u}_3 + \mathbf{u}_3(\mathbf{u}_1 \wedge \mathbf{u}_2)I_3] = -[(\mathbf{u}_1 \wedge \mathbf{u}_2)I_3] \cdot \mathbf{u}_3 = \left[(\mathbf{u}_1 \wedge \mathbf{u}_2)(I_3)^{-1}\right] \cdot \mathbf{u}_3\end{aligned}$$

where $(\mathbf{u}_1 \wedge \mathbf{u}_2)(I_3)^{-1}$ is a vector orthogonal to $span\{\mathbf{u}_1, \mathbf{u}_2\}$, because it is orthogonal to both $\mathbf{u}_1$ and $\mathbf{u}_2$, as

$$\left[(\mathbf{u}_1 \wedge \mathbf{u}_2)(I_3)^{-1}\right] \cdot \mathbf{u}_1 = (\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_1)(I_3)^{-1} = 0 \text{ and } \left[(\mathbf{u}_1 \wedge \mathbf{u}_2)(I_3)^{-1}\right] \cdot \mathbf{u}_2 = (\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_2)(I_3)^{-1} = 0$$

As a matter of fact, $(\mathbf{u}_1 \wedge \mathbf{u}_2)(I_3)^{-1}$ corresponds to the classical Gibbs and Heaviside cross product of $\mathbf{u}_1$ and $\mathbf{u}_2$, when $\mathbb{E}_3 = span\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is identified with $\mathbb{R}^3$.

4.2 | Coordinate-free expression of a hyperplane secant the graph of a three-variable function

Let us write the equation of a hyperplane secant the graph of a three-variable function $f : \Omega \subseteq \mathbb{E}_3 \to \mathbb{R}$ at four non-coplanar points of that graph. A hyperplane passing through the four non-coplanar points

$$(\mathbf{a}_i, f(\mathbf{a}_i)) = \left(\alpha_{i,1}\mathbf{e}_1 + \alpha_{i,2}\mathbf{e}_2 + \alpha_{i,3}\mathbf{e}_3, f(\mathbf{a}_i)\right) \in \mathbb{E}_3 \oplus \mathbb{R}$$

with $i = 1, 2, 3, 4$, can be represented by the Cartesian relation

$$\det\begin{pmatrix} \chi_1-\alpha_{1,1} & \chi_2-\alpha_{1,2} & \chi_3-\alpha_{1,3} & z-f(\mathbf{a}_1) \\ \alpha_{2,1}-\alpha_{1,1} & \alpha_{2,2}-\alpha_{1,2} & \alpha_{2,3}-\alpha_{1,3} & f(\mathbf{a}_2)-f(\mathbf{a}_1) \\ \alpha_{3,1}-\alpha_{1,1} & \alpha_{3,2}-\alpha_{1,2} & \alpha_{3,3}-\alpha_{1,3} & f(\mathbf{a}_3)-f(\mathbf{a}_1) \\ \alpha_{4,1}-\alpha_{1,1} & \alpha_{4,2}-\alpha_{1,2} & \alpha_{4,3}-\alpha_{1,3} & f(\mathbf{a}_4)-f(\mathbf{a}_1) \end{pmatrix} = 0 \tag{4.1}$$

between the real variables $\chi_1, \chi_2, \chi_3, z \in \mathbb{R}$. This determinant can be rewritten by a Laplace expansion as follows:

$$[z-f(\mathbf{a}_1)]\det\begin{pmatrix} \alpha_{2,1}-\alpha_{1,1} & \alpha_{2,2}-\alpha_{1,2} & \alpha_{2,3}-\alpha_{1,3} \\ \alpha_{3,1}-\alpha_{1,1} & \alpha_{3,2}-\alpha_{1,2} & \alpha_{3,3}-\alpha_{1,3} \\ \alpha_{4,1}-\alpha_{1,1} & \alpha_{4,2}-\alpha_{1,2} & \alpha_{4,3}-\alpha_{1,3} \end{pmatrix} - [f(\mathbf{a}_2)-f(\mathbf{a}_1)]\det\begin{pmatrix} \chi_1-\alpha_{1,1} & \chi_2-\alpha_{1,2} & \chi_3-\alpha_{1,3} \\ \alpha_{3,1}-\alpha_{1,1} & \alpha_{3,2}-\alpha_{1,2} & \alpha_{3,3}-\alpha_{1,3} \\ \alpha_{4,1}-\alpha_{1,1} & \alpha_{4,2}-\alpha_{1,2} & \alpha_{4,3}-\alpha_{1,3} \end{pmatrix} +$$
$$+[f(\mathbf{a}_3)-f(\mathbf{a}_1)]\det\begin{pmatrix} \chi_1-\alpha_{1,1} & \chi_2-\alpha_{1,2} & \chi_3-\alpha_{1,3} \\ \alpha_{2,1}-\alpha_{1,1} & \alpha_{2,2}-\alpha_{1,2} & \alpha_{2,3}-\alpha_{1,3} \\ \alpha_{4,1}-\alpha_{1,1} & \alpha_{4,2}-\alpha_{1,2} & \alpha_{4,3}-\alpha_{1,3} \end{pmatrix} - [f(\mathbf{a}_4)-f(\mathbf{a}_1)]\det\begin{pmatrix} \chi_1-\alpha_{1,1} & \chi_2-\alpha_{1,2} & \chi_3-\alpha_{1,3} \\ \alpha_{2,1}-\alpha_{1,1} & \alpha_{2,2}-\alpha_{1,2} & \alpha_{2,3}-\alpha_{1,3} \\ \alpha_{3,1}-\alpha_{1,1} & \alpha_{3,2}-\alpha_{1,2} & \alpha_{3,3}-\alpha_{1,3} \end{pmatrix}$$

Then, in $\mathbb{G}_3$, Equation (4.1) becomes

$$[z-f(\mathbf{a}_1)]\,[(\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)]\,(I_3)^{-1} - [f(\mathbf{a}_2)-f(\mathbf{a}_1)]\,[(\mathbf{x}-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)]\,(I_3)^{-1}+$$
$$+[f(\mathbf{a}_3)-f(\mathbf{a}_1)]\,[(\mathbf{x}-\mathbf{a}_1)\wedge(\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)]\,(I_3)^{-1} - [f(\mathbf{a}_4)-f(\mathbf{a}_1)]\,[(\mathbf{x}-\mathbf{a}_1)\wedge(\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)]\,(I_3)^{-1} = 0$$

being $\mathbf{x} = \chi_1\mathbf{e}_1 + \chi_2\mathbf{e}_2 + \chi_3\mathbf{e}_3 \in \mathbb{E}_3$, $(\mathbf{x}, z) \in \mathbb{E}_3 \oplus \mathbb{R}$. The foregoing relation is equivalent, in $\mathbb{G}_3$, to

$$\begin{aligned}[z-f(\mathbf{a}_1)]\,[(\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)] = {}& [f(\mathbf{a}_2)-f(\mathbf{a}_1)]\,[(\mathbf{x}-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)]+\\ &-[f(\mathbf{a}_3)-f(\mathbf{a}_1)]\,[(\mathbf{x}-\mathbf{a}_1)\wedge(\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)]+\\ &+[f(\mathbf{a}_4)-f(\mathbf{a}_1)]\,[(\mathbf{x}-\mathbf{a}_1)\wedge(\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)]\end{aligned}$$

Let us define

$$\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = (\mathbf{a}_2-\mathbf{a}_1)\wedge(\mathbf{a}_3-\mathbf{a}_1)\wedge(\mathbf{a}_4-\mathbf{a}_1)\ \left[\text{which is also equal to} -(\mathbf{a}_1-\mathbf{a}_2)\wedge(\mathbf{a}_2-\mathbf{a}_3)\wedge(\mathbf{a}_3-\mathbf{a}_4)\right].$$

We observe that

$$\tau_3 = \frac{1}{6}\det\begin{pmatrix} \alpha_{2,1}-\alpha_{1,1} & \alpha_{2,2}-\alpha_{1,2} & \alpha_{2,3}-\alpha_{1,3} \\ \alpha_{3,1}-\alpha_{1,1} & \alpha_{3,2}-\alpha_{1,2} & \alpha_{3,3}-\alpha_{1,3} \\ \alpha_{4,1}-\alpha_{1,1} & \alpha_{4,2}-\alpha_{1,2} & \alpha_{4,3}-\alpha_{1,3} \end{pmatrix} = \frac{1}{6}\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}(I_3)^{-1}$$

is the oriented volume of the tetrahedron having vertices $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$. So, we can write $\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = 6\tau_3 I_3$, and $\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)^{-1} = \frac{1}{6\tau}(I_3)^{-1}$. By denoting $\Delta f_{(\mathbf{a}_i)} = f(\mathbf{a}_i) - f(\mathbf{a}_1)$ and $\Delta\mathbf{a}_i = \mathbf{a}_i - \mathbf{a}_1$, when $i = 2, 3, 4$, the equation of the secant hyperplane (4.1) becomes

$$\begin{aligned} z &= f(\mathbf{a}_1) + \frac{1}{6\tau_3}\left\{\Delta f_{(\mathbf{a}_2)}(\mathbf{x}-\mathbf{a}_1)\wedge\Delta\mathbf{a}_3\wedge\Delta\mathbf{a}_4 - \Delta f_{(\mathbf{a}_3)}(\mathbf{x}-\mathbf{a}_1)\wedge\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_4 + \Delta f_{(\mathbf{a}_4)}(\mathbf{x}-\mathbf{a}_1)\wedge\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_3\right\}(I_3)^{-1} \\ &= f(\mathbf{a}_1) + \frac{1}{6\tau_3}\left\{\Delta f_{(\mathbf{a}_2)}\Delta\mathbf{a}_3\wedge\Delta\mathbf{a}_4\wedge(\mathbf{x}-\mathbf{a}_1) - \Delta f_{(\mathbf{a}_3)}\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_4\wedge(\mathbf{x}-\mathbf{a}_1) + \Delta f_{(\mathbf{a}_4)}\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_3\wedge(\mathbf{x}-\mathbf{a}_1)\right\}(I_3)^{-1} \\ &= f(\mathbf{a}_1) + \frac{1}{6\tau_3}\left\{\Delta f_{(\mathbf{a}_2)}(\Delta\mathbf{a}_3\wedge\Delta\mathbf{a}_4)(I_3)^{-1} - \Delta f_{(\mathbf{a}_3)}(\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_4)(I_3)^{-1} + \Delta f_{(\mathbf{a}_4)}(\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_3)(I_3)^{-1}\right\}\cdot(\mathbf{x}-\mathbf{a}_1) \\ &= f(\mathbf{a}_1) + \frac{1}{6\tau_3}\left\{\left[\Delta f_{(\mathbf{a}_2)}\Delta\mathbf{a}_3\wedge\Delta\mathbf{a}_4 - \Delta f_{(\mathbf{a}_3)}\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_4 + \Delta f_{(\mathbf{a}_4)}\Delta\mathbf{a}_2\wedge\Delta\mathbf{a}_3\right](I_3)^{-1}\right\}\cdot(\mathbf{x}-\mathbf{a}_1) \end{aligned}$$

Thus, we have that the vector $\mathbf{r}_{(f,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} \in \mathbb{E}_3$, characterizing the equation of the hyperplane secant the graph of f as

$z = f(\mathbf{a}_1) + \mathbf{r}_{(f,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} \cdot (\mathbf{x} - \mathbf{a}_1)$, is

$$\mathbf{r}_{(f,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \left[\sum_{i=2}^{4} (-1)^i \Delta f_{(\mathbf{a}_i)} \Delta^i_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right] \left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)^{-1}$$

(where $\Delta^2_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \Delta\mathbf{a}_3 \wedge \Delta\mathbf{a}_4$, $\Delta^3_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \Delta\mathbf{a}_2 \wedge \Delta\mathbf{a}_4$, $\Delta^4_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \Delta\mathbf{a}_2 \wedge \Delta\mathbf{a}_3$), which is, in fact, the Clifford quotient between the multidifference $\mathbb{G}_3$-pseudo-vector

$$\Delta f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \Delta f_{(\mathbf{a}_2)} \Delta\mathbf{a}_3 \wedge \Delta\mathbf{a}_4 - \Delta f_{(\mathbf{a}_3)} \Delta\mathbf{a}_2 \wedge \Delta\mathbf{a}_4 + \Delta f_{(\mathbf{a}_4)} \Delta\mathbf{a}_2 \wedge \Delta\mathbf{a}_3 \in \mathbb{G}_{\binom{3}{2}}$$

and the $\mathbb{G}_3$-pseudo-scalar

$$\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \Delta\mathbf{a}_2 \wedge \Delta\mathbf{a}_3 \wedge \Delta\mathbf{a}_4 = 6\tau_3 I_3 \in \mathbb{G}_{\binom{3}{2}} \simeq \mathbb{R} I_3$$

4.3 | Mirroring vectors and points II

4.3.1 | Vector mirrored by a 2-dimensional linear subspace

We recall that to each pair of linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{E}_n$, we can associate the 2-dimensional linear subspace $span\{\mathbf{u}_1, \mathbf{u}_2\} \subseteq \mathbb{E}_n$. Moreover, $\mathbf{u}_1 \wedge \mathbf{u}_2$ is always a 2-blade. As a matter of fact, there always exists an orthogonal basis $\{\mathbf{g}_1, \mathbf{g}_2\}$ of $span\{\mathbf{u}_1, \mathbf{u}_2\}$, and you can verify that $\mathbf{u}_1 \wedge \mathbf{u}_2$ is a nonzero multiple of the geometric product $\mathbf{g}_1\mathbf{g}_2$. We recall that

- the square of every 2-blade is a nonzero scalar,
- every 2-blade B is invertible in $\mathbb{G}_n$, and $B^{-1} = \frac{1}{B^2} B$.

Let us recall that, if $\mathbf{v} \in \mathbb{E}_n$, and $B = \mathbf{u}_1 \wedge \mathbf{u}_2$ is a 2-blade, then

$$\mathbf{v} = \mathbf{v}BB^{-1} = (\mathbf{v}B)B^{-1} = (\mathbf{v}\circ B + \mathbf{v} \curlywedge B)B^{-1} = (\mathbf{v}\circ B)B^{-1} + (\mathbf{v} \curlywedge B)B^{-1} \tag{4.2}$$

where

$$\begin{aligned}
\mathbf{v}\circ B &= \frac{1}{2}(\mathbf{v}B - B\mathbf{v}) = -B\circ\mathbf{v} \in \mathbb{G}_{\binom{n}{1}} \\
\mathbf{v} \curlywedge B &= \frac{1}{2}(\mathbf{v}B + B\mathbf{v}) = B \curlywedge \mathbf{v} \in \mathbb{G}_{\binom{n}{3}} \\
\mathbf{v}(\mathbf{u}_1 \wedge \mathbf{u}_2) &= \mathbf{v} \wedge \mathbf{u}_1 \wedge \mathbf{u}_2 \\
\mathbb{G}_{\binom{n}{k}} &= \begin{cases} \mathbb{R} & \text{if } k = 0 \\ span\{\mathbf{e}_{i_1} \cdots \mathbf{e}_{i_k}\}_{1 \le i_1 < \cdots < i_k \le n} & \text{if } 1 \le k \le n \end{cases}
\end{aligned}$$

Remark 17. Elements of $\mathbb{G}_{\binom{n}{k}}$ are called "k-vectors." Notice that the dimension of $\mathbb{G}_{\binom{n}{k}}$ is the binomial coefficient $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$. Moreover, $\mathbb{G}_n = \bigoplus_{k=0}^{n} \mathbb{G}_{\binom{n}{k}}$.

Remark 18. We recall that the foregoing operations "$\circ$" and "$\curlywedge$" can be extended[23] to k-blades $B = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k = \bigwedge_{j=1}^{k} \mathbf{u}_j \in \mathbb{G}_{\binom{n}{k}}$ where

$$\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k = \frac{1}{k!} \sum_{\sigma \in S_k} \epsilon_\sigma \mathbf{u}_{\sigma_1} \cdots \mathbf{u}_{\sigma_k}$$

[23] See, for example, [23] or [24].

($\mathcal{S}_k$ being the group of all permutations of $\{1, \dots, k\}$, $\epsilon_\sigma \in \{-1, 1\}$ the parity of permutation σ), as follows

$$\mathbf{v}\circ B = \frac{1}{2}\left(\mathbf{v}B - (-1)^k B\mathbf{v}\right) = (-1)^{k+1} B\circ\mathbf{v} \in \mathbb{G}_{\binom{n}{k-1}}$$
$$\mathbf{v} \curlywedge B = \frac{1}{2}\left(\mathbf{v}B + (-1)^k B\mathbf{v}\right) = (-1)^{k} B \curlywedge \mathbf{v} \in \mathbb{G}_{\binom{n}{k+1}}$$

We also recall that

$$\mathbf{v}\circ(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k) = \sum_{i=1}^{k} (-1)^{i+1}(\mathbf{v}\cdot\mathbf{u}_i) \bigwedge_{\substack{j=1\\ j\neq i}}^{k} \mathbf{u}_j$$
$$\mathbf{v} \curlywedge (\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k) = \mathbf{v} \wedge \mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_k$$

So, we have the following

$$B\mathbf{v} = 2(B\circ\mathbf{v}) + (-1)^k \mathbf{v}B = (B\circ\mathbf{v}) + (B \curlywedge \mathbf{v})$$

Remark 19. Notice that if $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{E}_n$, then $\mathbf{u}\circ\mathbf{v} = \mathbf{u}\cdot\mathbf{v}$ and $\mathbf{u}\mathbf{v} = \mathbf{u}\wedge\mathbf{v}$.

Remark 20. The operations "$\circ$" and "$\curlywedge$" are particular cases of more general operations between blades. More precisely, if H is a h-blade, and K is a k-blade, then

$$H\circ K = \frac{1}{2}\left(HK - (-1)^{hk} KH\right) = (-1)^{hk+1} K\circ H \text{ is called "graded commutator"}$$
$$H \curlywedge K = \frac{1}{2}\left(HK + (-1)^{k} KH\right) = (-1)^{hk} K \curlywedge H \text{ is called "graded anti-commutator."}$$

Such operations can then be extended, by linearity, to linear combinations of blades, that is, to every element of the geometric algebra $\mathbb{G}_n$.

Remark 21. If $B = \mathbf{u}_1 \wedge \mathbf{u}_2$ is a 2-blade, then $(\mathbf{v} \curlywedge B)B^{-1}$ is a vector that is orthogonal both to $\mathbf{u}_1$ and $\mathbf{u}_2$, that is, to all 2-dimensional $span\{\mathbf{u}_1, \mathbf{u}_2\}$. As a matter of fact, for $i = 1, 2$, we have that

$$2\left[(\mathbf{v} \curlywedge B)B^{-1}\right] \cdot \mathbf{u}_i = (\mathbf{v} \curlywedge B)B^{-1}\mathbf{u}_i + \mathbf{u}_i(\mathbf{v} \curlywedge B)B^{-1}$$

You can verify that $B\mathbf{u}_i = -B\mathbf{u}_i$ (for $i = 1, 2$). So we have that

$$2\left[(\mathbf{v} \curlywedge B)B^{-1}\right] \cdot \mathbf{u}_i = \left[-(\mathbf{v} \curlywedge B)\mathbf{u}_i + \mathbf{u}_i(\mathbf{v} \curlywedge B)\right] B^{-1}$$

As $\mathbf{v} \curlywedge B = \mathbf{v} \wedge \mathbf{u}_1 \wedge \mathbf{u}_2$ is either zero or a 3-blade, then we can write

$$4\left[(\mathbf{v} \curlywedge B)B^{-1}\right] \cdot \mathbf{u}_i = \left[\mathbf{u}_i \curlywedge (\mathbf{v} \curlywedge B)\right] B^{-1} = (\mathbf{u}_i \wedge \mathbf{v} \wedge \mathbf{u}_1 \wedge \mathbf{u}_2)\, B^{-1} = 0$$

In a similar way, you can verify that for every vector $\mathbf{v} \in \mathbb{E}_n$, we have that

$$\mathbf{v}_{\|} = (\mathbf{v}\circ B)B^{-1} \in span\{\mathbf{u}_1, \mathbf{u}_2\}$$

So, relation (4.2) corresponds to the orthogonal decomposition of vector $\mathbf{v}$ with respect to the 2-dimensional linear subspace $span\{\mathbf{u}_1, \mathbf{u}_2\} = \mathcal{L}_{(\mathbf{0},\mathbf{u}_1,\mathbf{u}_2)}$.

Then, the vector $\hat{\mathbf{v}}$, mirrored of vector $\mathbf{v}$ by the 2-dimensional linear subspace $span\{\mathbf{u}_1, \mathbf{u}_2\}$, can be written as

$$\hat{\mathbf{v}} = \mathbf{v}_{\|} - \mathbf{v}_{\perp} = (\mathbf{v}\circ B)B^{-1} - (\mathbf{v} \curlywedge B)B^{-1} = \left[(\mathbf{v}\circ B) - (\mathbf{v} \curlywedge B)\right] B^{-1} = \left[-(B\circ\mathbf{v}) - (B \curlywedge \mathbf{v})\right] B^{-1}$$
$$= -\left[(B\circ\mathbf{v}) + (B \curlywedge \mathbf{v})\right] B^{-1} = -B\mathbf{v}B^{-1} = -\left[2(B\circ\mathbf{v}) + \mathbf{v}B\right] B^{-1} = -2(B\circ\mathbf{v})B^{-1} - \mathbf{v}$$

Moreover, $|\hat{\mathbf{v}}| = |\mathbf{v}|$; as a matter of fact,

$$|\hat{\mathbf{v}}|^2 = \hat{\mathbf{v}}\hat{\mathbf{v}} = B\mathbf{v}B^{-1}B\mathbf{v}B^{-1} = B\mathbf{v}\mathbf{v}B^{-1} = |\mathbf{v}|^2 BB^{-1} = |\mathbf{v}|^2$$

Remark 22. Notice that $\hat{\mathbf{v}}$ is in $span\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}\}$.

4.3.2 | Point mirrored by a plane in $\mathbb{E}_n$ (with $n \geq 3$)

Given four non-coplanar points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ in $\mathbb{E}_n$ (with $n \geq 3$), we mirror the point $\mathbf{a}_1$ through the plane $\mathcal{L}_{(\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}$ generated by the points $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$. We denote $\overline{\mathbf{a}_1}$ the mirrored point.

We can express $\overline{\mathbf{a}_1}$ by computing in $\mathbb{G}_n$ the vector $\hat{\mathbf{v}}$, obtained by mirroring vector $\mathbf{v} = \mathbf{a}_1 - \mathbf{a}_2$ by the 2-blade $L_1 = L_{(\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = (\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_2)$.

More precisely, the mirrored point $\overline{\mathbf{a}_1}$ can be computed by using the geometric Clifford product in $\mathbb{G}_n$ as follows:

$$\begin{aligned}\overline{\mathbf{a}_1} &= \mathbf{a}_2 - L_1(\mathbf{a}_1 - \mathbf{a}_2)(L_1)^{-1} = \mathbf{a}_2 - 2\,[L_1 \circ (\mathbf{a}_1 - \mathbf{a}_2)]\,(L_1)^{-1} - (\mathbf{a}_1 - \mathbf{a}_2)\\ &= 2\mathbf{a}_2 - 2\,[L_1 \circ (\mathbf{a}_1 - \mathbf{a}_2)]\,(L_1)^{-1} - \mathbf{a}_1 \in span\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}\end{aligned}$$

Thus,

$$\begin{aligned}\overline{\mathbf{a}_1} - \mathbf{a}_1 &= 2(\mathbf{a}_2 - \mathbf{a}_1) - 2\,[L_1 \circ (\mathbf{a}_1 - \mathbf{a}_2)]\,(L_1)^{-1} = 2(\mathbf{a}_2 - \mathbf{a}_1) + 2\,[(\mathbf{a}_1 - \mathbf{a}_2) \circ L_1]\,(L_1)^{-1}\\ &= 2(\mathbf{a}_2 - \mathbf{a}_1)L_1(L_1)^{-1} - 2\,[(\mathbf{a}_2 - \mathbf{a}_1) \circ L_1]\,(L_1)^{-1} = 2\,\{(\mathbf{a}_2 - \mathbf{a}_1)L_1 - [(\mathbf{a}_2 - \mathbf{a}_1) \circ L_1]\}\,(L_1)^{-1}\\ &= 2\,[(\mathbf{a}_2 - \mathbf{a}_1) \mathbin{\wedge\!\!\!\wedge} L_1]\,(L_1)^{-1} = 2\,[(\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_2)]\,[(\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_2)]^{-1}\\ &= 2\,[(\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_2)]\,[(\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_3)]^{-1}\end{aligned}$$

and you can verify that $|\overline{\mathbf{a}_1} - \mathbf{a}_2| = |\mathbf{a}_1 - \mathbf{a}_2|$.

Remark 23. By using Remark 21, you can verify that the vector $\overline{\mathbf{a}_1} - \mathbf{a}_1$ is orthogonal to both $\mathbf{a}_2 - \mathbf{a}_3$ and $\mathbf{a}_3 - \mathbf{a}_4$.

Then, we mirror the point $\mathbf{a}_2$ through the line $\mathcal{L}_{(\mathbf{a}_3,\mathbf{a}_4)}$, obtaining $\overline{\mathbf{a}_2}$ such that

$$\overline{\mathbf{a}_2} - \mathbf{a}_2 = 2\,[(\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_3)]\,(\mathbf{a}_4 - \mathbf{a}_3)^{-1} \in span\{\mathbf{a}_3 - \mathbf{a}_2\,,\, \mathbf{a}_4 - \mathbf{a}_3\}$$

as shown in Section 3.3.2. Thus, we have now that vectors $\overline{\mathbf{a}_1} - \mathbf{a}_1$, $\overline{\mathbf{a}_2} - \mathbf{a}_2$, and $\mathbf{a}_3 - \mathbf{a}_4$ are mutually orthogonal. Moreover,

$$\begin{aligned}\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right) \wedge (\mathbf{a}_4 - \mathbf{a}_3) &= \left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)(\mathbf{a}_4 - \mathbf{a}_3) = 2\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)[(\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_3)]\,(\mathbf{a}_4 - \mathbf{a}_3)^{-1}(\mathbf{a}_4 - \mathbf{a}_3)\\ &= 2\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)[(\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_3)] = 4(\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_2) \wedge (\mathbf{a}_4 - \mathbf{a}_2)\\ &= 4(\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_1) \wedge (\mathbf{a}_4 - \mathbf{a}_1) = 4\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\end{aligned}$$

4.4 | Convergence of the mean secant hyperplane plane to the tangent hyperplane I

Definition 2. Let us define the "mean multi-difference $\mathbb{G}_{\binom{3}{2}}$-pseudo-vector,"

$$\begin{aligned}\overline{\Delta}f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \frac{1}{4}\Big\{&\left[f\left(\overline{\mathbf{a}_1}\right) - f(\mathbf{a}_1)\right]\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right) \wedge (\mathbf{a}_4 - \mathbf{a}_3) - \left[f\left(\overline{\mathbf{a}_2}\right) - f(\mathbf{a}_2)\right]\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge (\mathbf{a}_4 - \mathbf{a}_3) +\\ &+ [f(\mathbf{a}_4) - f(\mathbf{a}_3)]\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)\Big\} \in \mathbb{G}_{\binom{3}{2}}\end{aligned}$$

Remark 24. The term "mean" in the foregoing definition is due to the fact that

$$\overline{\Delta}f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \frac{1}{4}\left\{\Delta f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} - \Delta f_{(\overline{\mathbf{a}_1},\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} - \Delta f_{(\mathbf{a}_1,\overline{\mathbf{a}_2},\mathbf{a}_3,\mathbf{a}_4)} + \Delta f_{(\overline{\mathbf{a}_1},\overline{\mathbf{a}_2},\mathbf{a}_3,\mathbf{a}_4)}\right\}$$

as you can verify.

Remark 25. The mean multidifference $\mathbb{G}_{\binom{3}{2}}$-pseudo-vector can also be written as

$$\overline{\Delta} f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \frac{1}{4}\left\{\left[f\left(\overline{\mathbf{a}_1}\right) - f(\mathbf{a}_1)\right]\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)(\mathbf{a}_4 - \mathbf{a}_3) - \left[f\left(\overline{\mathbf{a}_2}\right) - f(\mathbf{a}_2)\right]\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)(\mathbf{a}_4 - \mathbf{a}_3) + \right.$$
$$\left. + [f(\mathbf{a}_4) - f(\mathbf{a}_3)]\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)\right\}$$

because vectors $\overline{\mathbf{a}_1} - \mathbf{a}_1$, $\overline{\mathbf{a}_2} - \mathbf{a}_2$, and $\mathbf{a}_4 - \mathbf{a}_3$ are mutually orthogonal.

Proposition 2.

$$\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \frac{f\left(\overline{\mathbf{a}_1}\right) - f(\mathbf{a}_1)}{\overline{\mathbf{a}_1} - \mathbf{a}_1} + \frac{f\left(\overline{\mathbf{a}_2}\right) - f(\mathbf{a}_2)}{\overline{\mathbf{a}_2} - \mathbf{a}_2} + \frac{f(\mathbf{a}_4) - f(\mathbf{a}_3)}{\mathbf{a}_4 - \mathbf{a}_3}$$

Proof of Proposition 2. From Remark 23, we have that $\overline{\mathbf{a}_1} - \mathbf{a}_1$, $\overline{\mathbf{a}_2} - \mathbf{a}_2$, $\mathbf{a}_4 - \mathbf{a}_3$ are mutually orthogonal, and $\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)(\mathbf{a}_4 - \mathbf{a}_3) = 4\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}$. So, we can write

$$\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} = \frac{1}{4}\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right) \wedge (\mathbf{a}_4 - \mathbf{a}_3) = \frac{1}{4}\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)(\mathbf{a}_4 - \mathbf{a}_3)$$

$$\begin{aligned}\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)^{-1} &= 4(\mathbf{a}_4 - \mathbf{a}_3)^{-1}\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)^{-1}\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)^{-1} = -4(\mathbf{a}_4 - \mathbf{a}_3)^{-1}\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)^{-1}\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)^{-1} \\ &= 4\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)^{-1}\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)^{-1}(\mathbf{a}_4 - \mathbf{a}_3)^{-1}\end{aligned} \tag{4.3}$$

Then, by Remark 25, we can write

$$\begin{aligned}\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)} &= \left(\overline{\Delta} f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)^{-1} = \\ &= \left[f\left(\overline{\mathbf{a}_1}\right) - f(\mathbf{a}_1)\right]\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)^{-1} + \left[f\left(\overline{\mathbf{a}_2}\right) - f(\mathbf{a}_2)\right]\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)^{-1} + [f(\mathbf{a}_4) - f(\mathbf{a}_3)](\mathbf{a}_4 - \mathbf{a}_3)^{-1}\end{aligned}$$

□

Theorem 2. *If the function* $f : \Omega \subseteq \mathbb{E}_3 \to \mathbb{R}$ *is strongly differentiable at* $\mathbf{x}_0$ *(a point internal to* Ω*), then*

$$\lim_{\substack{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)\to(\mathbf{x}_0,\mathbf{x}_0,\mathbf{x}_0,\mathbf{x}_0) \\ \mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4 \ \text{not coplanar}}} \left(\overline{\Delta} f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)^{-1} = \nabla f(\mathbf{x}_0)$$

Remark 26. The foregoing result state that the "mean secant hyperplane"

$$z = f(\mathbf{a}_1) + \overline{\mathbf{r}}_{f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}} \cdot (\mathbf{x} - \mathbf{a}_1)$$

where $\overline{\mathbf{r}}_{f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}} = \left(\overline{\Delta} f_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)^{-1}$, always converges to the tangent hyperplane

$$z = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$$

as the non-degenerate tetrahedron having vertices $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ contracts to the point $\mathbf{x}_0$.

Proof of Theorem 2. Let us observe that

$$\begin{aligned}\nabla f(\mathbf{x}_0)\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right) &= \nabla f(\mathbf{x}_0)\circ\left(\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}\right)\\ &= \frac{1}{4}\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\right]\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\wedge(\mathbf{a}_4-\mathbf{a}_3)-\frac{1}{4}\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\right]\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\wedge(\mathbf{a}_4-\mathbf{a}_3)+\\ &\quad+\frac{1}{4}\left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_4-\mathbf{a}_3)\right]\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\wedge\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\\ &= \frac{1}{4}\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\right]\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)(\mathbf{a}_4-\mathbf{a}_3)-\frac{1}{4}\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\right]\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)(\mathbf{a}_4-\mathbf{a}_3)+\\ &\quad+\frac{1}{4}\left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_4-\mathbf{a}_3)\right]\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\end{aligned}$$

By relations (4.3), we can write

$$\begin{aligned}\nabla f(\mathbf{x}_0) &= \nabla f(\mathbf{x}_0)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)\left(\Delta_{(\mathbf{a},\mathbf{b},\mathbf{c})}\right)^{-1}\\ &= \left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\right]\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)^{-1}+\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\right]\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)^{-1}+\left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_4-\mathbf{a}_3)\right](\mathbf{a}_4-\mathbf{a}_3)^{-1}\\ &= \frac{\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)}{\overline{\mathbf{a}_1}-\mathbf{a}_1}+\frac{\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)}{\overline{\mathbf{a}_2}-\mathbf{a}_2}+\frac{\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_4-\mathbf{a}_3)}{\mathbf{a}_4-\mathbf{a}_3}\end{aligned}$$

So, by Proposition 2, we can write

$$\begin{aligned}\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}-\nabla f(\mathbf{x}_0) &= \frac{f\left(\overline{\mathbf{a}_1}\right)-f(\mathbf{a}_1)-\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\right]}{\overline{\mathbf{a}_1}-\mathbf{a}_1}+\\ &\quad+\frac{f\left(\overline{\mathbf{a}_2}\right)-f(\mathbf{a}_2)-\left[\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\right]}{\overline{\mathbf{a}_2}-\mathbf{a}_2}+\frac{f(\mathbf{a}_4)-f(\mathbf{a}_3)-\left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_4-\mathbf{a}_3)\right]}{\mathbf{a}_4-\mathbf{a}_3}\end{aligned}$$

Thus,

$$\begin{aligned}\left|\overline{\mathbf{r}}_{f(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}-\nabla f(\mathbf{x}_0)\right| &\le \frac{\left|f\left(\overline{\mathbf{a}_1}\right)-f(\mathbf{a}_1)-\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\right|}{\left|\overline{\mathbf{a}_1}-\mathbf{a}_1\right|}+\\ &\quad+\frac{\left|f\left(\overline{\mathbf{a}_2}\right)-f(\mathbf{a}_2)-\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_2}-\mathbf{a}_2\right)\right|}{\left|\overline{\mathbf{a}_2}-\mathbf{a}_2\right|}+\frac{\left|f(\mathbf{a}_4)-f(\mathbf{a}_3)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_4-\mathbf{a}_3)\right|}{\left|\mathbf{a}_4-\mathbf{a}_3\right|}\end{aligned}$$

As f is strongly differentiable at $\mathbf{x}_0$, we know that, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that if $|\mathbf{u}-\mathbf{x}_0| < \delta_\epsilon$ and $|\mathbf{v}-\mathbf{x}_0| < \delta_\epsilon$, then

$$|f(\mathbf{u})-f(\mathbf{v})-\nabla f(\mathbf{x}_0)\cdot(\mathbf{u}-\mathbf{v})| \le \epsilon|\mathbf{u}-\mathbf{v}|$$

So, if we choose $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$ non-collinear, and such that

$$|\mathbf{a}_i-\mathbf{x}_0| < \frac{1}{3}\left(\delta_{\frac{\epsilon}{3}}\right), \quad \text{for } i = 1,2,3,4$$

then

$$\left|\overline{\mathbf{r}}_{f(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}-\nabla f(\mathbf{x}_0)\right| \le \epsilon$$

as, for $i = 1,2,3$, we have

- $|\mathbf{a}_i-\mathbf{a}_{i+1}| \le |\mathbf{a}_i-\mathbf{x}_0|+|\mathbf{a}_{i+1}-\mathbf{x}_0| < \frac{2}{3}\left(\delta_{\frac{\epsilon}{3}}\right)$
- $|\overline{\mathbf{a}_i}-\mathbf{x}_0| \le |\overline{\mathbf{a}_i}-\mathbf{a}_{i+1}|+|\mathbf{a}_{i+1}-\mathbf{x}_0| = |\mathbf{a}_i-\mathbf{a}_{i+1}|+|\mathbf{a}_{i+1}-\mathbf{x}_0| < \delta_{\frac{\epsilon}{3}}$

□

Remark 27. The key property that provide the convergence of the mean secant hyperplane is the following chain of identities

$$\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right)\left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)(\mathbf{a}_4 - \mathbf{a}_3) = \left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right) \wedge (\mathbf{a}_4 - \mathbf{a}_3) = 4\Delta_{(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3,\mathbf{a}_4)}$$

implied by the orthogonality properties of the constructed mirrored points based on the four non-coplanar points $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$.

5 | THE CASE OF A MULTIVARIABLE FUNCTION

5.1 | A $k \times k$ determinant as a Clifford quotient and as a scalar product

The determinant of a $k \times k$ real matrix

$$\begin{pmatrix} \mu_{1,1} & \dots & \mu_{1,k} \\ \vdots & \ddots & \vdots \\ \mu_{k,1} & \dots & \mu_{k,k} \end{pmatrix}$$

can be written as a coordinate-free Clifford quotient in $\mathbb{G}_n$ (with $n \geq k$). Let us fix any ordered set $\mathbf{e}_1, \dots, \mathbf{e}_k$ of mutually orthonormal vectors in $\mathbb{E}_n$, and let us consider

$$\mathbf{u}_i = \sum_{j=1}^{k} \mu_{i,j}\, \mathbf{e}_j$$

with $i = 1, \dots, k$, then you can verify that

$$(\mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_k)(I_k)^{-1} = \det \begin{pmatrix} \mu_{1,1} & \dots & \mu_{1,k} \\ \vdots & \ddots & \vdots \\ \mu_{k,1} & \dots & \mu_{k,k} \end{pmatrix}$$

because

$$\mathbf{u}_{\sigma_1} \wedge \dots \wedge \mathbf{u}_{\sigma_k} = \epsilon_\sigma\, \mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_k$$

for each permutation $\sigma \in \mathcal{S}_k$ of the set $\{1, \dots, k\}$, having parity $\epsilon_\sigma \in \{-1, 1\}$, where

$$I_k = \mathbf{e}_1 \dots \mathbf{e}_k = \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_k, \text{ so that} (I_k)^{-1} = \mathbf{e}_k \dots \mathbf{e}_1 = (-1)^{\frac{k(k-1)}{2}} I_k$$

Remark 28. As I_2 and I_3 before, also I_k does not depend on the particular orthonormal basis of $span\{\mathbf{e}_1, \dots, \mathbf{e}_k\} = \mathbb{E}_k \subseteq \mathbb{E}_n$ to define it but only on the orientation of that basis. More precisely, if $\{\mathbf{g}_1, \dots, \mathbf{g}_k\}$ is any other orthonormal basis of $\mathbb{E}_k$, then $\mathbf{g}_1 \wedge \dots \wedge \mathbf{g}_k = \mathbf{g}_1 \dots \mathbf{g}_k$ is equal to I_k or $-I_k$. That is why I_k is called an "orientation" of $\mathbb{E}_k$.

Thus, a $k \times k$ determinant can be considered as the Clifford ratio between the two "k-blades" $\mathbf{u}_1 \wedge \dots \wedge \mathbf{u}_k$ and I_k (a "k-blade" being the geometric product of k nonzero and mutually orthogonal vectors). Those elements can also be called "$\mathbb{G}_k$-pseudo-scalars," as $\mathbb{G}_k$ is generated by $\mathbb{E}_k = span\{\mathbf{e}_1, \dots, \mathbf{e}_k\}$, and can be interpreted as oriented hyper-volumes in $\mathbb{E}_k$.

In general, if $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is an orthonormal basis for $\mathbb{E}_n$ and

$$V = \sum_{1 \leq i_1 < \dots < i_k \leq n} v_{i_1, \dots, i_k} \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_k} \in \mathbb{G}_{\binom{n}{k}}$$

(such elements in $\mathbb{G}_n$ are also called the "k-vectors"), then

$$\begin{aligned} VI_n &= \sum_{1\le i_1<\cdots<i_k\le n} v_{i_1,\dots,i_k}\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k}\mathbf{e_1}\cdots\mathbf{e_n} = (-1)^{n-1}\sum_{1\le i_1<\cdots<i_k\le n} v_{i_1,\dots,i_k}\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_{k-1}}\mathbf{e_1}\cdots\mathbf{e_n}\mathbf{e}_{i_k} \\ &= (-1)^{(n-1)k}\sum_{1\le i_1<\cdots<i_k\le n} v_{i_1,\dots,i_k}\mathbf{e_1}\cdots\mathbf{e_n}\mathbf{e}_{i_1}\cdots\mathbf{e}_{i_k} \in \mathbb{G}_{\binom{n}{n-k}} \\ &= (-1)^{(n-1)k} I_n V \end{aligned}$$

that is why, when $k = n-1$, the elements in $\mathbb{G}_{\binom{n}{n-1}}$ are also called "$\mathbb{G}_n$-pseudo-vectors": Geometric multiplication by I_n establishes a duality between elements of $\mathbb{G}_{\binom{n}{n-1}}$ and vectors in $\mathbb{E}_n = \mathbb{G}_{\binom{n}{1}}$. So, we have that $VI_n = (-1)^{(n-1)^2} I_n V = (-1)^{n-1} I_n V$, for each $\mathbb{G}_n$-pseudo-vector $V \in \mathbb{G}_{\binom{n}{n-1}}$.

The foregoing properties allows us to write a $k\times k$ determinant also as a scalar product

$$\begin{aligned} \det\begin{pmatrix} \mu_{1,1} & \dots & \mu_{1,k} \\ \vdots & \ddots & \vdots \\ \mu_{k,1} & \dots & \mu_{k,k} \end{pmatrix} &= (\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k)(I_k)^{-1} = (-1)^{\frac{k(k-1)}{2}}(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_k)I_k = (-1)^{\frac{k(k-1)}{2}}\left[(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})\mathbin{\bar\wedge}\mathbf{u}_k\right]I_k \\ &= \frac{(-1)^{\frac{k(k-1)}{2}}}{2}\left[(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})\mathbf{u}_k + (-1)^{k-1}\mathbf{u}_k(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})\right]I_k \\ &= \frac{(-1)^{\frac{k(k-1)}{2}}}{2}\left[(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})\mathbf{u}_k I_k + (-1)^{k-1}\mathbf{u}_k(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})I_k\right] \\ &= (-1)^{k-1}\frac{(-1)^{\frac{k(k-1)}{2}}}{2}\left[(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})I_k\mathbf{u}_k + \mathbf{u}_k(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})I_k\right] \\ &= (-1)^{k-1}\left[(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})(I_k)^{-1}\right]\cdot\mathbf{u}_k \end{aligned}$$

We can also verify that the vector $(\mathbf{u}_1\wedge\dots\wedge\mathbf{u}_{k-1})I_k$ is a vector orthogonal to $span\{\mathbf{u}_1,\dots,\mathbf{u}_{k-1}\}$, because it is orthogonal to each $\mathbf{u}_i$ (when $i = 1,\dots,k-1$), as you can verify that

$$\left[(\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1})(I_k)^{-1}\right]\cdot\mathbf{u}_i = (\mathbf{u}_1\wedge\cdots\wedge\mathbf{u}_{k-1}\wedge\mathbf{u}_i)(I_k)^{-1} = 0$$

5.2 | Coordinate-free expression of a hyperplane secant the graph of a multivariable function

Let us write the equation of a hyperplane secant the graph of a multivariable function $f : \Omega\subseteq\mathbb{E}_n\to\mathbb{R}$ at $n+1$ points of that graph non being on a same $(n-1)$-dimensional hyperplane. If $\mathbf{e}_1,\dots,\mathbf{e}_n$ is an orthonormal basis for $\mathbb{E}_n$, a hyperplane passing through such $n+1$ points

$$(\mathbf{a}_i, f(\mathbf{a}_i)) = \left(\sum_{j=1}^n \alpha_{i,j}\mathbf{e}_j, f(\mathbf{a}_i)\right) \in \mathbb{E}_n\oplus\mathbb{R}$$

with $i = 1,\dots,n+1$, can be represented by the Cartesian relation

$$\det\begin{pmatrix} \chi_1-\alpha_{1,1} & \dots & \chi_n-\alpha_{1,n} & z-f(\mathbf{a}_1) \\ \alpha_{2,1}-\alpha_{1,1} & \dots & \alpha_{2,n}-\alpha_{1,n} & f(\mathbf{a}_2)-f(\mathbf{a}_1) \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{n+1,1}-\alpha_{1,1} & \dots & \alpha_{n+1,n}-\alpha_{1,n} & f(\mathbf{a}_{n+1})-f(\mathbf{a}_1) \end{pmatrix} = 0 \tag{5.1}$$

between the $n+1$ real variables $\chi_1, \dots, \chi_n, z \in \mathbb{R}$. This determinant can be rewritten by a Laplace expansion as follows:

$$[z - f(\mathbf{a}_1)] \det \begin{pmatrix} \alpha_{2,1} - \alpha_{1,1} & \dots & \alpha_{2,n} - \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n+1,1} - \alpha_{1,1} & \dots & \alpha_{n+1,n} - \alpha_{1,n} \end{pmatrix} - [f(\mathbf{a}_2) - f(\mathbf{a}_1)] \det \begin{pmatrix} \chi_1 - \alpha_{1,1} & \dots & \chi_n - \alpha_{1,n} \\ \alpha_{3,1} - \alpha_{1,1} & \dots & \alpha_{3,n} - \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n+1,1} - \alpha_{1,1} & \dots & \alpha_{n+1,n} - \alpha_{1,n} \end{pmatrix} +$$
$$+ [f(\mathbf{a}_3) - f(\mathbf{a}_1)] \det \begin{pmatrix} \chi_1 - \alpha_{1,1} & \dots & \chi_n - \alpha_{1,n} \\ \alpha_{2,1} - \alpha_{1,1} & \dots & \alpha_{2,n} - \alpha_{1,n} \\ \alpha_{4,1} - \alpha_{1,1} & \dots & \alpha_{4,n} - \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n+1,1} - \alpha_{1,1} & \dots & \alpha_{n+1,n} - \alpha_{1,n} \end{pmatrix} - \dots + (-1)^n [f(\mathbf{a}_{n+1}) - f(\mathbf{a}_1)] \det \begin{pmatrix} \chi_1 - \alpha_{1,1} & \dots & \chi_n - \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} - \alpha_{1,1} & \dots & \alpha_{n,n} - \alpha_{1,n} \end{pmatrix}$$

Then, in $\mathbb{G}_3$, Equation (5.1) becomes

$$\begin{aligned} &[z - f(\mathbf{a}_1)] [(\mathbf{a}_2 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1)] (I_n)^{-1} - [f(\mathbf{a}_2) - f(\mathbf{a}_1)] [(\mathbf{x} - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1)] (I_n)^{-1} + \\ &+ [f(\mathbf{a}_3) - f(\mathbf{a}_1)] [(\mathbf{x} - \mathbf{a}_1) \wedge (\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_4 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1)] (I_n)^{-1} - \dots + \\ &+ (-1)^n [f(\mathbf{a}_{n+1}) - f(\mathbf{a}_1)] [(\mathbf{x} - \mathbf{a}_1) \wedge (\mathbf{a}_2 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_n - \mathbf{a}_1)] (I_3)^{-1} = 0 \end{aligned}$$

being $\mathbf{x} = \sum_{i=1}^{n} \chi_i \mathbf{e}_i \in \mathbb{E}_n$, $(\mathbf{x}, z) \in \mathbb{E}_n \oplus \mathbb{R}$. The foregoing relation is equivalent, in $\mathbb{G}_3$, to

$$\begin{aligned} [z - f(\mathbf{a}_1)] [(\mathbf{a}_2 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1)] = {} & [f(\mathbf{a}_2) - f(\mathbf{a}_1)] [(\mathbf{x} - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1)] + \\ & - [f(\mathbf{a}_3) - f(\mathbf{a}_1)] [(\mathbf{x} - \mathbf{a}_1) \wedge (\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_4 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1)] + \\ & + \dots + (-1)^{n+1} [f(\mathbf{a}_{n+1}) - f(\mathbf{a}_1)] [(\mathbf{x} - \mathbf{a}_1) \wedge (\mathbf{a}_2 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_n - \mathbf{a}_1)] \end{aligned}$$

Let us define

$$\Delta_{(\mathbf{a}_1, \dots, \mathbf{a}_{n+1})} = (\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_1) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1) = \bigwedge_{i=2}^{n+1} (\mathbf{a}_i - \mathbf{a}_1)$$
$$\left[\text{which is also equal to } (-1)^n (\mathbf{a}_1 - \mathbf{a}_2) \wedge (\mathbf{a}_2 - \mathbf{a}_3) \wedge \dots \wedge (\mathbf{a}_n - \mathbf{a}_{n+1})\right]$$

We observe that

$$\tau_n = \frac{1}{n!} \det \begin{pmatrix} \alpha_{2,1} - \alpha_{1,1} & \dots & \alpha_{2,n} - \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n+1,1} - \alpha_{1,1} & \dots & \alpha_{n+1,n} - \alpha_{1,n} \end{pmatrix} = \frac{1}{n!} \Delta_{(\mathbf{a}_1, \dots, \mathbf{a}_{n+1})} (I_n)^{-1}$$

is the oriented hyper-volume of the simplex having vertices $\mathbf{a}_1, \dots, \mathbf{a}_n$, and $\mathbf{a}_{n+1}$. So, we can write $\Delta_{(\mathbf{a}_1, \dots, \mathbf{a}_{n+1})} = n! \tau_n I_n$, and $\left(\Delta_{(\mathbf{a}_1, \dots, \mathbf{a}_{n+1})}\right)^{-1} = \frac{1}{n! \tau_n} (I_n)^{-1}$. By denoting $\Delta f_{(\mathbf{a}_i)} = f(\mathbf{a}_i) - f(\mathbf{a}_1)$ and $\Delta \mathbf{a}_i = \mathbf{a}_i - \mathbf{a}_1$, when $i = 2, \dots, n+1$, the equation of the secant hyperplane (5.1) becomes

$$\begin{aligned} z = f(\mathbf{a}_1) + \frac{1}{n! \tau_n} \big\{ & \Delta f_{(\mathbf{a}_2)} (\mathbf{x} - \mathbf{a}_1) \wedge \Delta \mathbf{a}_3 \wedge \dots \wedge \Delta \mathbf{a}_{n+1} - \Delta f_{(\mathbf{a}_3)} (\mathbf{x} - \mathbf{a}_1) \wedge \Delta \mathbf{a}_2 \wedge \Delta \mathbf{a}_4 \wedge \dots \wedge \Delta \mathbf{a}_{n+1} + \dots + \\ & + (-1)^{n+1} \Delta f_{(\mathbf{a}_{n+1})} (\mathbf{x} - \mathbf{a}_1) \wedge \Delta \mathbf{a}_2 \wedge \dots \wedge \Delta \mathbf{a}_n \big\} (I_n)^{-1} \end{aligned}$$

If we define

$$\Delta^i_{(\mathbf{a}_1, \dots, \mathbf{a}_{n+1})} = \bigwedge_{\substack{j=2 \\ j \neq i}}^{n+1} \Delta \mathbf{a}_j = \Delta^i \in \mathbb{G}_{\binom{n}{n-1}}$$

the equation of the secant hyperplane (5.1) becomes

$$
\begin{aligned}
z &= f(\mathbf{a}_1) + \frac{1}{n!\tau_n}\left\{\Delta f_{(\mathbf{a}_2)}(\mathbf{x}-\mathbf{a}_1) \curlywedge \Delta^2 - \Delta f_{(\mathbf{a}_3)}(\mathbf{x}-\mathbf{a}_1) \curlywedge \Delta^3 + \cdots + \Delta f_{(\mathbf{a}_{n+1})}(\mathbf{x}-\mathbf{a}_1) \curlywedge \Delta^{n+1}\right\}(I_n)^{-1} \\
&= f(\mathbf{a}_1) + \frac{1}{n!\tau_n}\left\{(\mathbf{x}-\mathbf{a}_1) \curlywedge \left[\Delta f_{(\mathbf{a}_2)}\Delta^2 - \Delta f_{(\mathbf{a}_3)}\Delta^3 + \cdots + \Delta f_{(\mathbf{a}_{n+1})}\Delta^{n+1}\right]\right\}(I_n)^{-1} \\
&= f(\mathbf{a}_1) + \frac{(-1)^{n-1}}{n!\tau_n}\left\{\left[\Delta f_{(\mathbf{a}_2)}\Delta^2 - \Delta f_{(\mathbf{a}_3)}\Delta^3 + \cdots + \Delta f_{(\mathbf{a}_{n+1})}\Delta^{n+1}\right] \curlywedge (\mathbf{x}-\mathbf{a}_1)\right\}(I_n)^{-1} \\
&= f(\mathbf{a}_1) + \frac{1}{n!\tau_n}\left\{\left[\Delta f_{(\mathbf{a}_2)}\Delta^2 - \Delta f_{(\mathbf{a}_3)}\Delta^3 + \cdots + \Delta f_{(\mathbf{a}_{n+1})}\Delta^{n+1}\right](I_n)^{-1}\right\} \cdot (\mathbf{x}-\mathbf{a}_1) \\
&= f(\mathbf{a}_1) + \left\{\left[\Delta f_{(\mathbf{a}_2)}\Delta^2 - \Delta f_{(\mathbf{a}_3)}\Delta^3 + \cdots + \Delta f_{(\mathbf{a}_{n+1})}\Delta^{n+1}\right](\Delta_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})})^{-1}\right\} \cdot (\mathbf{x}-\mathbf{a}_1)
\end{aligned}
$$

Thus, we have that the vector $\mathbf{r}_{(f,\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} \in \mathbb{E}_n$, characterizing the equation of the hyperplane secant the graph of f as $z = f(\mathbf{a}_1) + \mathbf{r}_{(f,\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} \cdot (\mathbf{x}-\mathbf{a}_1)$, is

$$
\mathbf{r}_{(f,\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} = \left[\sum_{i=2}^{n+1} (-1)^i \Delta f_{(\mathbf{a}_i)} \Delta^i_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})}\right]\left(\Delta_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})}\right)^{-1}
$$

which is, in fact, the Clifford quotient between the multi-difference $\mathbb{G}_n$-pseudo-vector

$$
\Delta f_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} = \sum_{i=2}^{n+1} (-1)^i \Delta f_{(\mathbf{a}_i)} \Delta^i_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} \in \mathbb{G}_{\binom{n}{n-1}}
$$

and the $\mathbb{G}_n$-pseudo-scalar $\Delta_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} \in \mathbb{G}_{\binom{n}{n}} \simeq \mathbb{R}I_n$.

5.3 | Mirroring vectors and points III

Here, we simply iterate the process already followed in the two foregoing cases. More precisely, if $n > 3$, we consider $n+1$ points $\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}$ in $\mathbb{E}_n$ as vertices of a non-degenerate simplex (i.e., $\Delta_{(\mathbf{a}_1,\ldots,\mathbf{a}_{n+1})} \neq 0$), we construct $n-1$ mirrored points $\overline{\mathbf{a}_1}, \ldots, \overline{\mathbf{a}_{n-1}}$ in $\mathbb{E}_n$ in the following way:

- $\overline{\mathbf{a}_1}$ is obtained by mirroring point $\mathbf{a}_1$ through the hyperplane $\mathcal{L}_{(\mathbf{a}_2,\ldots,\mathbf{a}_{n+1})}$;
- $\overline{\mathbf{a}_2}$ is obtained by mirroring point $\mathbf{a}_2$ through the hyperplane $\mathcal{L}_{(\mathbf{a}_3,\ldots,\mathbf{a}_{n+1})}$;
- $\vdots$
- $\overline{\mathbf{a}_{n-2}}$ is obtained by mirroring point $\mathbf{a}_{n-2}$ through the plane $\mathcal{L}_{(\mathbf{a}_{n-1},\mathbf{a}_n,\mathbf{a}_{n+1})}$;
- $\overline{\mathbf{a}_{n-1}}$ is obtained by mirroring point $\mathbf{a}_{n-1}$ through the line $\mathcal{L}_{(\mathbf{a}_n,\mathbf{a}_{n+1})}$,

where $\mathcal{L}_{(\mathbf{v}_1,\ldots,\mathbf{v}_h)} = \{v_1\mathbf{v}_1 + \cdots + v_h\mathbf{v}_h \;:\; v_1,\cdots,v_h \in \mathbb{R},\, v_1 + \cdots + v_h = 1\}$ is the hyperplane passing through points $\mathbf{v}_1, \ldots, \mathbf{v}_h \in \mathbb{E}_n$.

All that points $\overline{\mathbf{a}_i}$ can be obtained by mirroring vectors $\mathbf{a}_i - \mathbf{a}_{i+1}$ through the $(n-i)$-dimensional subspace $span\{\mathbf{a}_{i+2} - \mathbf{a}_{i+1}, \ldots, \mathbf{a}_{n+1} - \mathbf{a}_{i+1}\}$, respectively, by using the corresponding $(n-i)$-blade

$$
L_i = (\mathbf{a}_{i+2} - \mathbf{a}_{i+1}) \wedge \cdots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_{i+1})
$$

as computed in the two following paragraphs.

5.3.1 | Vector mirrored by a k-dimensional linear subspace

We recall that to k linearly independent vectors $\mathbf{u}_1, \ldots \mathbf{u}_k \in \mathbb{E}_n$, we can associate the k-dimensional linear subspace $span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \mathbb{E}_n$. Moreover, $\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k$ is always a k-blade. As a matter of fact, there always exists an orthogonal basis $\{\mathbf{g}_1, \ldots, \mathbf{g}_k\}$ of $span\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$, and you can verify that $\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_k$ is a nonzero multiple of the geometric product $\mathbf{g}_1 \ldots \mathbf{g}_k$. We recall that

- the square of every k-blade is a nonzero scalar,

- every k-blade B is invertible in $\mathbb{G}_n$, and $B^{-1} = \frac{1}{B^2}B$.

Let us recall that, if $\mathbf{v} \in \mathbb{E}_n$, and $B = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ is a k-blade, then

$$\mathbf{v} = \mathbf{v}BB^{-1} = (\mathbf{v}B)B^{-1} = (\mathbf{v}\circ B + \mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1} = (\mathbf{v}\circ B)B^{-1} + (\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1} \tag{5.2}$$

where

$$\mathbf{v}\circ B = \frac{1}{2}\left(\mathbf{v}B - (-1)^k B\mathbf{v}\right) = (-1)^{k+1} B\circ\mathbf{v} \in \mathbb{G}_{\binom{n}{k-1}}$$
$$\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B = \frac{1}{2}\left(\mathbf{v}B + (-1)^k B\mathbf{v}\right) = (-1)^{k} B \mathbin{\wedge\!\!\!\wedge} \mathbf{v} \in \mathbb{G}_{\binom{n}{k+1}}$$

We also recall that

$$\mathbf{v}\circ(\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k) = \sum_{i=1}^{k} (-1)^{i+1} (\mathbf{v}\cdot\mathbf{u}_i) \bigwedge_{\substack{j=1\\ j\neq i}}^{k} \mathbf{u}_j \tag{5.3}$$

$$\mathbf{v} \mathbin{\wedge\!\!\!\wedge} (\mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k) = \mathbf{v} \wedge \mathbf{u}_1 \wedge \ \dots \ \wedge \mathbf{u}_k \tag{5.4}$$

So, we have that

$$B\mathbf{v} = 2(B\circ\mathbf{v}) + (-1)^k \mathbf{v}B = (B\circ\mathbf{v}) + (B \mathbin{\wedge\!\!\!\wedge} \mathbf{v})$$

Remark 29. If $B = \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ is a k-blade, then $(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1}$ is a vector that is orthogonal to all $\mathbf{u}_1, \dots, \mathbf{u}_k$, that is, to all k-dimensional $span\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$. As a matter of fact, for $i = 1, \dots, k$, we have that

$$2\left[(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1}\right]\cdot\mathbf{u}_i = (\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1}\mathbf{u}_i + \mathbf{u}_i(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1}$$

You can verify that $B\mathbf{u}_i = B\circ\mathbf{u}_i = (-1)^{k+1}\mathbf{u}_i\circ B = (-1)^{k+1} B\mathbf{u}_i$ (for $i = 1, \dots, k$). So we have that

$$2\left[(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1}\right]\cdot\mathbf{u}_i = \left[(-1)^{k+1}(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)\mathbf{u}_i + \mathbf{u}_i(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)\right]B^{-1}$$

As $\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B = \mathbf{v} \wedge \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k$ is either zero or a $(k+1)$-blade, then we can write

$$4\left[(\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1}\right]\cdot\mathbf{u}_i = \left[\mathbf{u}_i \mathbin{\wedge\!\!\!\wedge} (\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)\right]B^{-1} = (\mathbf{u}_i \wedge \mathbf{v} \wedge \mathbf{u}_1 \wedge \cdots \wedge \mathbf{u}_k)B^{-1} = 0$$

In a similar way, you can verify that for every vector $\mathbf{v} \in \mathbb{E}_n$, we have that

$$\mathbf{v}_{\|} = (\mathbf{v}\circ B)B^{-1} \in span\{\mathbf{u}_1, \cdots, \mathbf{u}_k\}$$

So, relation (5.2) corresponds to the orthogonal decomposition of vector $\mathbf{v}$ with respect to the k-dimensional linear subspace $span\{\mathbf{u}_1, \dots, \mathbf{u}_k\} = \mathcal{L}_{(\mathbf{0},\mathbf{u}_1,\dots,\mathbf{u}_k)}$.

Then, the vector $\hat{\mathbf{v}}$, mirrored of vector $\mathbf{v}$ by the k-dimensional linear subspace $span\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$, can be written as

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{v}_{\|} - \mathbf{v}_{\perp} = (\mathbf{v}\circ B)B^{-1} - (\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)B^{-1} = \left[(\mathbf{v}\circ B) - (\mathbf{v} \mathbin{\wedge\!\!\!\wedge} B)\right]B^{-1} = \left[(-1)^{k+1}(B\circ\mathbf{v}) - (-1)^k(B \mathbin{\wedge\!\!\!\wedge} \mathbf{v})\right]B^{-1}\\ &= (-1)^{k+1}\left[(B\circ\mathbf{v}) + (B \mathbin{\wedge\!\!\!\wedge} \mathbf{v})\right]B^{-1} = (-1)^{k+1}B\mathbf{v}B^{-1} = (-1)^{k+1}\left[2(B\circ\mathbf{v}) + (-1)^k\mathbf{v}B\right]B^{-1} = (-1)^{k+1}2(B\circ\mathbf{v})B^{-1} - \mathbf{v}\end{aligned}$$

Moreover, $|\hat{\mathbf{v}}| = |\mathbf{v}|$; as a matter of fact,

$$|\hat{\mathbf{v}}|^2 = \hat{\mathbf{v}}\hat{\mathbf{v}} = B\mathbf{v}B^{-1}B\mathbf{v}B^{-1} = B\mathbf{v}\mathbf{v}B^{-1} = |\mathbf{v}|^2BB^{-1} = |\mathbf{v}|^2$$

Remark 30. Notice that $\hat{\mathbf{v}}$ is in $span\{\mathbf{u}_1, \dots, \mathbf{u}_k, \mathbf{v}\}$.

5.3.2 Point mirrored by a multidimensional hyperplane in $\mathbb{E}_n$

Let $n > 3$, we consider $n+1$ points $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ in $\mathbb{E}_n$ such that $\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} \neq 0$.

While $1 \leq i \leq n-1$, we consider the a hyperplane $\mathcal{L}_{(\mathbf{a}_{i+1},\dots,\mathbf{a}_{n+1})}$ passing through $n+1-i$ points $\mathbf{a}_{i+1}, \dots, \mathbf{a}_{n+1}$. Then, we mirror the point $\mathbf{a}_i$ through that hyperplane $\mathcal{L}_{(\mathbf{a}_{i+1},\dots,\mathbf{a}_{n+1})}$. We denote $\overline{\mathbf{a}_i}$ the mirrored point.

We can express $\overline{\mathbf{a}_i}$ by computing in $\mathbb{G}_n$ the vector $\hat{\mathbf{v}}$, obtained by mirroring vector $\mathbf{v} = \mathbf{a}_i - \mathbf{a}_{i+1}$ by the $(n-i)$-blade

$$L_i = L_{(\mathbf{a}_{i+1},\dots,\mathbf{a}_{n+1})} = (\mathbf{a}_{i+2} - \mathbf{a}_{i+1}) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_{i+1}) = \bigwedge_{j=2}^{n+1-i} (\mathbf{a}_{i+j} - \mathbf{a}_{i+1})$$

for $i = 1, \dots, n-1$. More precisely, the mirrored point $\overline{\mathbf{a}_1}$ can be computed by using the geometric Clifford product in $\mathbb{G}_n$ as follows:

$$\begin{aligned}\overline{\mathbf{a}_i} &= \mathbf{a}_{i+1} + (-1)^{n-i+1} L_i(\mathbf{a}_i - \mathbf{a}_{i+1})(L_i)^{-1} = \mathbf{a}_{i+1} + (-1)^{n-i+1} 2\,[L_i \circ (\mathbf{a}_i - \mathbf{a}_{i+1})]\,(L_i)^{-1} - (\mathbf{a}_i - \mathbf{a}_{i+1}) \\ &= 2\mathbf{a}_{i+1} + (-1)^{n-i+1} 2\,[L_i \circ (\mathbf{a}_i - \mathbf{a}_{i+1})]\,(L_i)^{-1} - \mathbf{a}_i \in span\{\mathbf{a}_i, \mathbf{a}_{i+2}, \dots, \mathbf{a}_{n+1}\}\end{aligned}$$

Thus,

$$\begin{aligned}\overline{\mathbf{a}_i} - \mathbf{a}_i &= 2(\mathbf{a}_{i+1} - \mathbf{a}_i) + (-1)^{n-i+1} 2\,[L_i \circ (\mathbf{a}_i - \mathbf{a}_{i+1})]\,(L_i)^{-1} = 2(\mathbf{a}_{i+1} - \mathbf{a}_i) + 2\,[(\mathbf{a}_i - \mathbf{a}_{i+1}) \circ L_i]\,(L_i)^{-1} \\ &= 2(\mathbf{a}_{i+1} - \mathbf{a}_i) L_i (L_i)^{-1} - 2\,[(\mathbf{a}_{i+1} - \mathbf{a}_i) \circ L_i]\,(L_i)^{-1} = 2\,\{(\mathbf{a}_{i+1} - \mathbf{a}_i) L_i - [(\mathbf{a}_{i+1} - \mathbf{a}_i) \circ L_i]\}\,(L_i)^{-1} \\ &= 2\,[(\mathbf{a}_{i+1} - \mathbf{a}_i) \mathbin{\bar{\wedge}} L_i]\,(L_i)^{-1}\end{aligned}$$

and you can verify that $|\overline{\mathbf{a}_i} - \mathbf{a}_{i+1}| = |\mathbf{a}_i - \mathbf{a}_{i+1}|$. Let us prove a key property in the construction of the mirrored points.

Lemma 2. *By keeping the notations and hypothesis of this section, we have that*

$$(\mathbf{a}_{i+1} - \mathbf{a}_i) \mathbin{\bar{\wedge}} L_i = L_{i-1}$$

for each $i = 2, \dots, n-1$.

Proof of Lemma 2.

$$\begin{aligned}(\mathbf{a}_{i+1} - \mathbf{a}_i) \mathbin{\bar{\wedge}} L_i &= (\mathbf{a}_{i+1} - \mathbf{a}_i) \wedge (\mathbf{a}_{i+2} - \mathbf{a}_{i+1}) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_{i+1}) \\ &= (\mathbf{a}_{i+1} - \mathbf{a}_i) \wedge (\mathbf{a}_{i+2} - \mathbf{a}_i + \mathbf{a}_i - \mathbf{a}_{i+1}) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_i + \mathbf{a}_i - \mathbf{a}_{i+1}) \\ &= (\mathbf{a}_{i+1} - \mathbf{a}_i) \wedge (\mathbf{a}_{i+2} - \mathbf{a}_i) \wedge \dots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_i) = L_{i-1}\end{aligned}$$

Remark 31. By using Remark 29, you can also verify that each vector $\overline{\mathbf{a}_i} - \mathbf{a}_i$ is orthogonal to all $\mathbf{a}_{i+2} - \mathbf{a}_{i+1}, \dots, \mathbf{a}_{n+1} - \mathbf{a}_{i+1}$.

At the end of that process of mirroring the $n-1$ points $\mathbf{a}_1, \dots, \mathbf{a}_{n-1}$, we have that vectors

$$\overline{\mathbf{a}_1} - \mathbf{a}_1, \dots, \overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}, \text{ and } \mathbf{a}_{n+1} - \mathbf{a}_n$$

are mutually orthogonal. Moreover, the following key property holds.

Proposition 3. *By keeping the notations and hypothesis of this section, we have that*

$$\left[\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \dots \wedge \left(\overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}\right)\right] \mathbin{\bar{\wedge}} (\mathbf{a}_{n+1} - \mathbf{a}_n) = 2^{n-1} \Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}$$

Proof of Proposition 3. By Lemma 2, we can write $\overline{\mathbf{a}_i} - \mathbf{a}_i = 2L_{i-1}(L_i)^{-1}$. So, by the foregoing orthogonality property, we can arrive at a telescopic product

$$\begin{aligned}
&\left[\left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right) \wedge \left(\overline{\mathbf{a}_3} - \mathbf{a}_3\right) \wedge \cdots \wedge \left(\overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}\right)\right] \curlywedge (\mathbf{a}_{n+1} - \mathbf{a}_n) = \\
&= \left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right) \wedge \left(\overline{\mathbf{a}_3} - \mathbf{a}_3\right) \wedge \cdots \wedge \left(\overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}\right) \wedge (\mathbf{a}_{n+1} - \mathbf{a}_n) \\
&= \left(\overline{\mathbf{a}_2} - \mathbf{a}_2\right)\left(\overline{\mathbf{a}_3} - \mathbf{a}_3\right) \cdots \left(\overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}\right)(\mathbf{a}_{n+1} - \mathbf{a}_n) = \\
&= 2^{n-1}\left[(\mathbf{a}_2 - \mathbf{a}_1) \curlywedge L_1\right](L_1)^{-1} L_1 (L_2)^{-1} L_2 (L_3)^{-1} \cdots L_{n-2}(L_{n-1})^{-1}(\mathbf{a}_{n+1} - \mathbf{a}_n) \\
&= 2^{n-1}\left[(\mathbf{a}_2 - \mathbf{a}_1) \curlywedge L_1\right](L_{n-1})^{-1}(\mathbf{a}_{n+1} - \mathbf{a}_n) = 2^{n-1}\left[(\mathbf{a}_2 - \mathbf{a}_1) \curlywedge L_1\right](\mathbf{a}_{n+1} - \mathbf{a}_n)^{-1}(\mathbf{a}_{n+1} - \mathbf{a}_n) \\
&= 2^{n-1}(\mathbf{a}_2 - \mathbf{a}_1) \curlywedge L_1 = 2^{n-1}(\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_2) \wedge \cdots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_2) \\
&= 2^{n-1}(\mathbf{a}_2 - \mathbf{a}_1) \wedge (\mathbf{a}_3 - \mathbf{a}_1) \wedge \cdots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1) = 2^{n-1}\Delta_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})}
\end{aligned}$$

□

5.4 | Convergence of the mean secant hyperplane to the tangent hyperplane II

Let us consider the $n+1$ points $\mathbf{a}_1, \ldots, \mathbf{a}_{n+1}$ in $\mathbb{E}_n$ as in the foregoing paragraphs.

Definition 3. Let us define the following blades:

$$\Delta = \Delta_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} = (\mathbf{a}_2 - \mathbf{a}_1) \wedge \ldots \wedge (\mathbf{a}_{n+1} - \mathbf{a}_1) = \bigwedge_{i=2}^{n+1} (\mathbf{a}_i - \mathbf{a}_1) \in \mathbb{G}_{\binom{n}{n}} \simeq \mathbb{R} I_n$$

$$\overline{\Delta} = \overline{\Delta}_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} = \left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \wedge \ldots \wedge \left(\overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}\right) = \bigwedge_{i=1}^{n-1} \left(\overline{\mathbf{a}_i} - \mathbf{a}_i\right) \in \mathbb{G}_{\binom{n}{n-1}}$$

$$\overline{\Delta^i} = \overline{\Delta^i}_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} = \left(\bigwedge_{\substack{j=1 \\ j \neq i}}^{n-1} \left(\overline{\mathbf{a}_j} - \mathbf{a}_j\right) \right) \curlywedge (\mathbf{a}_{n+1} - \mathbf{a}_n) \in \mathbb{G}_{\binom{n}{n-1}}$$

where $i = 1, \ldots, n-1$.

Then, we can define the "mean multidifference $\mathbb{G}_{\binom{n}{n-1}}$-pseudo-vector,"

$$\overline{\Delta} f_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} = \frac{1}{2^{n-1}} \sum_{i=1}^{n-1} (-1)^{i+1} \left[f(\overline{\mathbf{a}_i}) - f(\mathbf{a}_i)\right] \overline{\Delta^i}_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} + \frac{(-1)^{n+1}}{2^{n-1}} \left[f(\mathbf{a}_{n+1}) - f(\mathbf{a}_n)\right] \overline{\Delta}_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})}$$

Remark 32. Contrary to the pseudo-scalar Δ, the foregoing blades $\overline{\Delta}$ and $\overline{\Delta^i}$ can always be written as geometric products

$$\overline{\Delta} = \overline{\Delta}_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} = \left(\overline{\mathbf{a}_1} - \mathbf{a}_1\right) \ldots \left(\overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}\right) = \prod_{i=1}^{n-1} \left(\overline{\mathbf{a}_i} - \mathbf{a}_i\right)$$

$$\overline{\Delta^i} = \overline{\Delta^i}_{(\mathbf{a}_1, \ldots, \mathbf{a}_{n+1})} = \left(\prod_{\substack{j=1 \\ j \neq i}}^{n-1} \left(\overline{\mathbf{a}_j} - \mathbf{a}_j\right) \right) (\mathbf{a}_{n+1} - \mathbf{a}_n)$$

(where $i = 1, \ldots, n-1$), because vectors $\overline{\mathbf{a}_1} - \mathbf{a}_1, \ldots, \overline{\mathbf{a}_{n-1}} - \mathbf{a}_{n-1}$ and $\mathbf{a}_{n+1} - \mathbf{a}_n$ are mutually orthogonal. These are the key properties that allow us to prove the next general theorem.

Theorem 3. *If the function $f : \Omega \subseteq \mathbb{E}_n \to \mathbb{R}$ is strongly differentiable at $\mathbf{x}_0$ (a point internal to Ω), then*

$$\lim_{\substack{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})\to(\mathbf{x_0},\dots,\mathbf{x_0})\in\mathbb{E}_n^{n+1}\\ \Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\neq 0}} \left(\overline{\Delta} f_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1} = \nabla f(\mathbf{x}_0)$$

Remark 33. The foregoing result states that the "mean secant hyperplane"

$$z = f(\mathbf{a}_1) + \overline{\mathbf{r}}_{(f,\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\cdot(\mathbf{x}-\mathbf{a}_1)$$

where $\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} = \left(\overline{\Delta} f_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1}$, always converges to the tangent hyperplane

$$z = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)$$

as the non-degenerate simplex having vertices $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ contracts to the point $\mathbf{x}_0$.

Proposition 4.

$$\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} = \left(\sum_{i=1}^{n-1}\frac{f(\overline{\mathbf{a}_i})-f(\mathbf{a}_i)}{\overline{\mathbf{a}_i}-\mathbf{a}_i}\right) + \frac{f(\mathbf{a}_{n+1})-f(\mathbf{a}_n)}{\mathbf{a}_{n+1}-\mathbf{a}_n}$$

Proof of Proposition 4. Let us recall, from Remark 31, that vectors $\overline{\mathbf{a}_1}-\mathbf{a}_1, \dots, \overline{\mathbf{a}_{n-1}}-\mathbf{a}_{n-1}, \mathbf{a}_{n+1}-\mathbf{a}_n$ are mutually orthogonal. So, by Proposition 3, we can write

$$\overline{\Delta}_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} \curlywedge (\mathbf{a}_{n+1}-\mathbf{a}_n) = 2^{n-1}\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}$$

$$\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} = \frac{1}{2^{n-1}}\overline{\Delta}_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} \curlywedge (\mathbf{a}_{n+1}-\mathbf{a}_n) = \frac{1}{2^{n-1}}\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)\dots\left(\overline{\mathbf{a}_{n-1}}-\mathbf{a}_{n-1}\right)(\mathbf{a}_{n+1}-\mathbf{a}_n)$$

$$\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1} = 2^{n-1}(\mathbf{a}_{n+1}-\mathbf{a}_n)^{-1}\left(\overline{\mathbf{a}_{n-1}}-\mathbf{a}_{n-1}\right)^{-1}\cdots\left(\overline{\mathbf{a}_1}-\mathbf{a}_1\right)^{-1}$$

Then,

$$\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} = \left(\overline{\Delta} f_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1} =$$

$$= \left(\sum_{i=1}^{n-1}(-1)^{i+1}\left[f(\overline{\mathbf{a}_i})-f(\mathbf{a}_i)\right]\overline{\Delta}^i_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1} + (-1)^{n+1}\left[f(\mathbf{a}_{n+1})-f(\mathbf{a}_n)\right]\overline{\Delta}_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1}$$

$$= \left(\sum_{i=1}^{n-1}\left[f(\overline{\mathbf{a}_i})-f(\mathbf{a}_i)\right]\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)^{-1}\right) + \left[f(\mathbf{a}_{n+1})-f(\mathbf{a}_n)\right](\mathbf{a}_{n+1}-\mathbf{a}_n)^{-1}$$

as you can verify that

$$\overline{\Delta}\Delta^{-1} = (-1)^{n+1}2^{n-1}(\mathbf{a}_{n+1}-\mathbf{a}_n)^{-1} \text{ and, } \overline{\Delta}^i\Delta^{-1} = (-1)^{i+1}2^{n-1}\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)^{-1} \quad \text{for } i = 1\dots, n-1$$

□

Proof of Theorem 3. Let us recall that $\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}$ is a pseudo-scalar; so, by Equation (5.3), we have that

$$\nabla f(\mathbf{x}_0)\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} = \nabla f(\mathbf{x}_0)\circ\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})} = \frac{1}{2^{n-1}}\left[\sum_{i=1}^{n-1}(-1)^{i+1}\left(\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)\right)\overline{\Delta}^i + (-1)^{n+1}\left(\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_{n+1}-\mathbf{a}_n)\right)\overline{\Delta}\right]$$

and we can write

$$\begin{aligned}
\nabla f(\mathbf{x}_0) &= \nabla f(\mathbf{x}_0)\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1} = \left(\nabla f(\mathbf{x}_0)\circ\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)\left(\Delta_{(\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}\right)^{-1}\\
&= \frac{1}{2^{n-1}}\left[\sum_{i=1}^{n-1}(-1)^{i+1}\left(\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)\right)\overline{\Delta^i}+(-1)^{n+1}\left(\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_{n+1}-\mathbf{a}_n)\right)\overline{\Delta}\right]\Delta^{-1}\\
&= \left(\sum_{i=1}^{n-1}\left(\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)\right)\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)^{-1}\right)+\left[\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_{n+1}-\mathbf{a}_n)\right](\mathbf{a}_{n+1}-\mathbf{a}_n)^{-1}\\
&= \left(\sum_{i=1}^{n-1}\frac{\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)}{\overline{\mathbf{a}_i}-\mathbf{a}_i}\right)+\frac{\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_{n+1}-\mathbf{a}_n)}{\mathbf{a}_{n+1}-\mathbf{a}_n}
\end{aligned}$$

So,

$$\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}-\nabla f(\mathbf{x}_0)=\left(\sum_{i=1}^{n-1}\frac{f(\overline{\mathbf{a}_i})-f(\mathbf{a}_i)-\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)}{\overline{\mathbf{a}_i}-\mathbf{a}_i}\right)+\frac{f(\mathbf{a}_{n+1})-f(\mathbf{a}_n)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_{n+1}-\mathbf{a}_n)}{\mathbf{a}_{n+1}-\mathbf{a}_n}$$

and

$$\left|\overline{\mathbf{r}}_{(f,\mathbf{a}_1,\dots,\mathbf{a}_{n+1})}-\nabla f(\mathbf{x}_0)\right|\leq\sum_{i=1}^{n-1}\frac{\left|f(\overline{\mathbf{a}_i})-f(\mathbf{a}_i)-\nabla f(\mathbf{x}_0)\cdot\left(\overline{\mathbf{a}_i}-\mathbf{a}_i\right)\right|}{\left|\overline{\mathbf{a}_i}-\mathbf{a}_i\right|}+\frac{|f(\mathbf{a}_{n+1})-f(\mathbf{a}_n)-\nabla f(\mathbf{x}_0)\cdot(\mathbf{a}_{n+1}-\mathbf{a}_n)|}{|\mathbf{a}_{n+1}-\mathbf{a}_n|}$$

As f is strongly differentiable at $\mathbf{x}_0$, we know that, given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that if $|\mathbf{u}-\mathbf{x}_0| < \delta_\epsilon$ and $|\mathbf{v}-\mathbf{x}_0| < \delta_\epsilon$, then

$$|f(\mathbf{u})-f(\mathbf{v})-\nabla f(\mathbf{x}_0)\cdot(\mathbf{u}-\mathbf{v})|\leq\epsilon|\mathbf{u}-\mathbf{v}|$$

So, if we choose $\mathbf{a}_1, \dots, \mathbf{a}_{n+1}$ non-collinear, and such that

$$|\mathbf{a}_i-\mathbf{x}_0|<\frac{1}{3}\left(\delta_{\frac{\epsilon}{n}}\right),\quad\text{for } i=1,\dots,n+1$$

then

$$\left|\overline{\mathbf{r}}_{(f,\mathbf{a},\mathbf{b},\mathbf{c})}-\nabla f(\mathbf{x}_0)\right|\leq\epsilon$$

as, for $i = 1, \dots, n-1$, we have

- $|\mathbf{a}_i-\mathbf{a}_{i+1}|\leq|\mathbf{a}_i-\mathbf{x}_0|+|\mathbf{a}_{i+1}-\mathbf{x}_0|<\frac{2}{3}\left(\delta_{\frac{\epsilon}{n}}\right)$,
- $|\overline{\mathbf{a}_i}-\mathbf{x}_0|\leq|\overline{\mathbf{a}_i}-\mathbf{a}_{i+1}|+|\mathbf{a}_{i+1}-\mathbf{x}_0|=|\mathbf{a}_i-\mathbf{a}_{i+1}|+|\mathbf{a}_{i+1}-\mathbf{x}_0|<\delta_{\frac{\epsilon}{3}}$ □

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CONFLICT OF INTEREST STATEMENT

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ORCID

Paolo Roselli https://orcid.org/0000-0003-3834-9358

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