Chapter 2. The Layer Model

Summary

This is an algebraic calculation of the effect of an infrared absorber, a pane of glass essentially, on the mean temperature of the surface of the earth. By solving the energy budgets of the earth's surface and the pane of glass, the reader can see how the pane of glass traps outgoing IR light, leading to a warming of the surface. The layer model is not an accurate, detailed model suitable for a global warming forecast, but the principal of the greenhouse effect cannot be understood without understanding this model.

The Bare-Rock Layer Model

The temperature of the surface of the earth is controlled by the ways that energy comes in from the sun and shines back out to space as infrared. The sun shines a lot of light, because the temperature at the visible surface of the sun is high and therefore the energy flux $I = \varepsilon \sigma T^4$ is a large number. Sunlight strikes the earth and deposits some of its energy into the form of vibrations and other bouncings-around of the molecules of the earth. Neither the earth nor the sun are perfect blackbodies, but they are both almost blackbodies, as are most solids and liquids. (Gases are terrible blackbodies, as we will learn in the next chapter). The earth radiates heat to space in the form of infrared light. Earth light is much lower frequency and lower energy than sun light.

We are going to construct a simple model of the temperature of the earth. The word **model** is used quite a bit in scientific discussion, to mean a fairly wide variety of things. Sometimes the word is synonymous with "theory" or "idea", such as the Standard Model of Particle Physics. For doctors, a "model system" might be a mouse that has some disease that resembles a disease that human patients get. They can experiment on the mouse rather than experimenting on people. In climate science, models are used in two different ways. One way is to make forecasts. For this purpose, a model should be as realistic as possible, and should capture or include all of the processes that might be relevant in nature. This is typically a mathematical model implemented on a computer, although there's a nifty physical model of San Francisco Bay you should check out if you're ever in the neighborhood. Once such a model has been constructed, a climate scientist can perform what-if experiments on it that could never be done to the real world, to determine how sensitive the climate would be to changes in the brightness of the sun or properties of the atmosphere, for example.

The simple model that we are going to construct here is not intended for making predictions, but is rather intended to be a toy system that we can learn from. The model will demonstrate how the greenhouse effect works by stripping away lots of other aspects of the real world that would certainly be important for predicting climate change in the next century or the weather next week, but make the climate system more complicated and therefore more difficult to understand. The model we are going to explore is called the **layer model**. Understanding the layer model will not equip us to make detailed forecasts of future climate, but one cannot understand the workings of the real climate system without first understanding the layer model.

The layer model makes a few assumptions. One is that the amount of energy coming into the planet from sunlight is equal to the amount of energy leaving the earth as infrared. The real world may be out of energy balance for a little while or over some region, but the layer model is always exactly in balance. We want to balance the energy budget by requiring that the **outgoing energy flux F_{out}** must equal the **incoming energy flux,**

$$F_{in} = F_{out}$$

Let's begin with incoming sunlight. The **intensity of incoming sunlight I_{in}** at the average distance from the sun to the earth is about 1350 W / m^2. We'll consider the Watts part of this quantity first, then next the m^2 part. If you've ever seen Venus glowing in the twilight sky you know that some of the incoming visible light shines back out again as visible light. Venus' brightness is not blackbody radiation; it is hot on Venus but not hot enough to shine white-hot. This is **reflected** light. When light is reflected, its energy is not converted to vibrational energy of molecules in the earth and then re-radiated according to the blackbody emission spectrum of the planet. It just bounces back out to space. For the purposes of the layer model, it is as if the energy had never arrived on earth at all. The fraction of a planet's incoming visible light that is reflected back to space is called the planet's **albedo** and is given the symbol α (greek letter alpha). Snow, ice, and clouds are very reflective, and tend to increase a planet's albedo. The albedo of bright Venus is high, 70%, because of a thick layer of sulfuric-acid clouds in the Venusian atmosphere, and low, 0.15, for Mars because of a lack of clouds on that planet. Earth's albedo of about 0.3 depends on cloudiness and sea ice cover, which might change with changing climate.

Incoming solar energy that is not reflected is assumed to be absorbed into vibrational energy of molecules of the earth. Using a present-day Earthly albedo of 0.3, we can calculate that the intensity of sunlight that is absorbed by the earth is 1350 W / m$^{2\cdot}$ $(1 - \alpha)$ = 1000 W / m^2.

What about the area, the m^2 on the bottom of that fraction? If we want to get the total incoming flux for the whole planet, in units of W instead of W/m^2, we need to multiply by a factor of area,

$$F_{in}\left[W\right] = I\left[\frac{W}{m^2}\right] \cdot A\left[m^2\right]$$

What area shall we use? Sun shines on half of the surface of the earth at any one time, but the light is weak and wan on some parts of the earth, near dawn or dusk or in high latitudes, but much more intense near the equator at noon. The difference in intensity is caused by the angle of the incoming sunlight, not because the sunlight, measured head-on at the top of the atmosphere, is much different between low and high latitudes (Figure 2-1). How then to add up all the weak fluxes and the strong fluxes on the earth to find the total amount of energy that the earth is absorbing?

There's a nifty trick. Measure the size of the shadow. The area we are looking for is that of a circle, not a sphere. The area is

$$A\left[m^2\right] = \pi \; r_{earth}^2$$

Putting these together, the total incoming flux of energy to a planet by solar radiation is

$$F_{in} = \pi \; r_{earth}^2 \; (1 - \alpha) \; I_{in}$$

Our first construction of the layer model will have no atmosphere, only a bare rock sphere in space. A real bare rock in space, such as the moon or Mercury, is incredibly hot on the bright side and cold in the dark. The differences are much more extreme than they are on Earth or Venus where heat is carried by fluid atmospheres. Nevertheless, we are trying to find a single value for the temperature of the earth, to go along with a single value for each of the heat fluxes F_{in} and F_{out}. The real world is not all the same temperature, but we're going to ignore that in the layer model. The heat fluxes F_{in} and F_{out} may not balance each other in the real world, either, but they do in the layer model.

The rate at which the earth radiates energy to space is given by the Stefan-Boltzmann equation,

$$F_{out} = A \; \varepsilon \; \sigma \; T_{earth}^4$$

As we did for solar energy, we are here converting intensity I to total energy flux F by multiplying by an area A. What area is appropriate this time? Incoming sunlight is different from outgoing earthlight in that sunlight is all traveling in one direction, whereas earthlight leaves earth in all directions (Figure 2-2). Therefore the area over which the earth radiates energy to space is simply the area of the sphere, which is given by

$$A_{sphere} = 4 \; \pi \; r_{earth}^2$$

Therefore the total outgoing energy flux from a planet by blackbody radiation is

$$F_{out} = 4 \; \pi \; r_{earth}^2 \; \varepsilon \; \sigma \; T_{earth}^4$$

The layer model assumes that the energy fluxes in and out balance each other

$$F_{out} = F_{in}$$

which means that we can construct an equation from the "pieces" of F_{out} and F_{in} which looks like this:

$$4\ \pi\ r_{earth}^2\ \varepsilon\ \sigma\ T_{earth}^4 = \pi\ r_{earth}^2\ (1-\alpha)\ I_{in}$$

This equation tells us how the temperature of the earth ought to respond to factors such as the intensity of sunlight and the albedo of the earth.

The first thing to notice is that a few terms appear in common on both sides of the equation, which means that we can cancel them by dividing both sides of the equation by those factors. These are π and r_{earth}^2. Before we toss these factors aside, however, we need to stop for another general science education tangent break and talk about π.

Box: π and transcendental numbers

The number π is one of a very special type of numbers called transcendental numbers. The simplest numbers are counting numbers; 1,2,3, etc. If we add digits after the decimal place, such as 2.7 and 3.24, we get rational numbers. A rational number can run on forever, such as the decimal expression for the fraction 1/9, which is 0.11111… with an infinite number of ones following the decimal point. They go on forever, but there is some repetition in the sequence of digits, likes "ones forever" in this example. Then there are the irrational numbers, such as the square root of two $\left(\sqrt{2}\right)$*, which can be approximated as 1.41421356…. These numbers go on forever like the rational numbers may do, but they never repeat. Pythagoras thought that numbers like this were an abomination, and their discovery was kept a closely guarded secret. There is no precise way to write the exact decimal value of* $\sqrt{2}$*, but we can get a handle on the number using geometry. In this case (and how Pythagoras came across* $\sqrt{2}$*), we can use the Pythagorean theorem to construct a line segment which is exactly* $\sqrt{2}$ *in length (Figure 2-3).*

Finally, the most elusive and exalted of them all are the transcendental numbers like π. The decimal expression of π, 3.1415927… goes on forever as many rational and all irrational numbers do. The digits after the decimal never repeat, like the irrational numbers. Transcendental numbers differ from irrational numbers is that there is no way using a compass and straightedge to construct a straight line whose length is π. Of course, it's easy to construct a curved line of length π by drawing a half of a circle using a compass, but that doesn't count; we are looking for a straight line. This endeavor has been named "squaring the circle" and has been the focus of much devotion among amateur mathematicians and kooks over the centuries, along with perpetual motion machines and the fountain of youth. A resident of my home state of Indiana managed to convince the Indiana State Legislature in 1897 that he had squared the circle to discover that the true value for π. He persuaded them to recognize the truth of his proof in *state law, in exchange for allowing them to use his value of π without paying royalties or license fees. Hoosier generosity!*

Before we move on, reflect for a moment that the number π is a transcendental number, one of the most elusive of all numbers. With modern computers we can approximate π as closely as we wish, but we can never write the value of π exactly. The exact area of the shadow of the earth, $\pi\ r_{earth}^2$*, will be forever out of our reach, as will the exact area of the surface of the sphere,* $4\ \pi\ r_{earth}^2$*. Yet we can write the ratio of these two unreachable quantities, with the simple number 4. Perhaps this is an indication that I don't get out enough, but I find this beautifully fascinating.*

OK, end of tangent. Time to do our first calculation of the temperature of the earth (Figure 2-4). Eliminating the factors of $\pi\ r_{earth}^2$, and dividing by 4 on both sides so that we are left with units of Watts per area of the earth's surface, we get

$$\varepsilon\ \sigma\ T_{earth}^4 = \frac{(1-\alpha)\ I_{in}}{4}$$

We know everything here except the T_{earth}. If we rearrange the equation to put what we know on the right hand side, what we don't on the left, we get

$$T_{earth} = \sqrt[4]{\frac{(1-\alpha)\ I_{in}}{4\ \varepsilon\ \sigma}} \qquad (4)$$

What we have constructed is a relationship between a number of crucial climate quantities. Changes in solar intensity such as the sunspot cycle or the Maunder Minimum (Chapter 10) may affect I_{in}. We shall see in the next chapter that greenhouse gases are extremely selective about the wavelengths of light that they absorb and emit; in other words they have complexities relating to their emissivity ε values. The albedo of the planet is very sensitive to ice and cloud cover, both of which might change with changing climate. All kinds of interesting possibilities.

If we calculate the temperature of the earth, we get a value of 255 K or about –15°C. This is too cold; the temperature range of earth's climate is –50° to about +35° C, but the average temperature, what we're calculating using the layer model, is closer to +15°C than –15°C. Table 2-1 gives the values we need to do the same calculation for Venus and Mars, along with the results of the calculation and the observed average temperatures. In all three cases, our calculation has erred on the side of too cold.

Table 2-1. The temperatures and albedos of the terrestrial planets.

	I_{solar}, W/m^2	α	T_{bare} K	$T_{observed}$ K	$T_{1\ layer}$ K
Venus	2600	71%	240	700	285
Earth	1350	33%	253	295	303
Mars	600	17%	216	240	259

The Layer Model With Greenhouse Effect

Our simple model is too cold because it lacks the greenhouse effect. We had no atmosphere on our planet; what we calculated was the temperature of a bare rock in space, like the moon. Of course the surface of the moon has a very different temperature on the sunlit side than it does in the shade, but if the incoming sunlight were spread out uniformly over the moon's surface, or if somehow the heat from one side of the planet conducted quickly to the other side, or if the planet rotated real fast, our calculation would be pretty good. But to get the temperature of the earth, Venus, and Mars, we need a greenhouse effect.

In keeping with the philosophy of the layer model, the atmosphere in the layer model is simple to the point of absurdity. The atmosphere in the layer model resembles a pane of glass suspended by magic above the ground Figure 2-5. Like glass, our atmosphere is transparent to visible light, so the incoming energy from the sun passes right through the atmosphere and is deposited on the planet surface, as before. The planet radiates energy as IR light according to $\varepsilon\ \sigma\ T_{ground}^4$, as before. In the IR range of light, we will assume that the atmosphere, like a pane of glass, acts as a blackbody, able to absorb and emit all frequencies of IR light. Therefore the energy flowing upward from the ground, in units of W/m^2 of the earth's surface, which we will call $I_{up,\ ground}$, is entirely absorbed by the atmospheric pane of glass. The atmosphere in turn radiates energy according to $\varepsilon\ \sigma\ T_{atmosphere}^4$. Because the pane of glass has two sides, a top side and a bottom side, it radiates energy both upward and downward, $I_{up,\ atmosphere}$ and $I_{down,\ atmosphere}$.

The layer model assumes that the energy budget is in steady state; energy in = energy out. This is true for any piece of the model, such as the atmosphere, just as it is for the planet as a whole. Therefore we can write an **energy budget for the atmosphere**, in units of Watts per area of the earth's surface, as

$$I_{up,\ atmosphere} + I_{down,\ atmosphere} = I_{up,\ ground}$$

or

$$2\ \varepsilon\ \sigma\ T_{atmosphere}^{4} = \varepsilon\ \sigma\ T_{ground}^{4}$$

The budget for the ground is different from before, because we now have heat flowing down from the atmosphere. The basic balance is

$$I_{out} = I_{in}$$

We can break these down into component fluxes

$$I_{up,\ ground} = I_{in,\ solar} + I_{down,\ atmosphere}$$

and then further dissect them into

$$\varepsilon\ \sigma\ T_{ground}^{4} = \frac{(1-\alpha)}{4} I_{solar} + \varepsilon\ \sigma\ T_{atmosphere}^{4}$$

Finally, we can also write a **budget for the earth overall** by drawing a boundary above the atmosphere and figuring that if energy gets across this line in, it must also be flowing across the line out at the same rate.

$$I_{up,\ atmosphere} = I_{in,\ solar}$$

The intensities are comprised of individual fluxes from the sun and from the atmosphere

$$\varepsilon\ \sigma\ T_{atmosphere}^{4} = \frac{(1-\alpha)}{4} I_{solar}$$

There is a solution to the layer model, for which all the budgets balance. We are looking for a pair of temperatures T_{ground} and $T_{atmosphere}$. Solving for T_{ground} and $T_{atmosphere}$ is a somewhat more complex problem algebraically than the bare-rock model with no atmosphere we solved above, but we can do it. We have two unknowns, but we appear to have three equations, the budgets for the atmosphere, for the ground, and for the earth overall. According to the rules of linear algebra, we can solve a problem if we have the same number of independent constraints (equations) as we have unknowns. If we have more constraints than unknowns, there might not be a solution. Can't get there from here. In our case, any two of the equations will suffice to pin down the two unknowns, and any third turns out to be just a combination of the other two. The budget equation for the earth overall, for example, is just the sum of the budget equations for the ground and the atmosphere (verify this for yourself). Therefore the third equation contains no new information that wasn't already contained in the first two, and we have a perfectly well-posed algebraic system, where the number of unknowns in just balanced by the number of independent constraints or equations.

So we are free to use any two of the three budget equations to solve for the unknowns T_{ground} and $T_{atmosphere}$. There are laborious ways to approach this problem, and there is an easy way. Shall we choose the easy way? OK. The easy way is to begin with the energy budget for the earth overall. This equation contains only one unknown, $T_{atmosphere}$. Come to think of it, this equation looks a lot like equation 4, describing the surface temperature of the bare planet model above. If we solve for $T_{atmosphere}$ here, we get the same answer as when we solved for $T_{bare\ earth}$. This is an important point, more than just a curiosity or an algebraic convenience. It tells us that the place in the earth system where the temperature is the most directly controlled by the rate of incoming solar energy is the temperature at the location that radiates to space. We will call this temperature the **skin temperature** of the earth.

What about temperatures below the skin, in this case T_{ground}? Now that we know that the outermost temperature, $T_{atmosphere}$, is equal to the skin temperature, we can plug that into the budget equation for the atmosphere to see that

$$2\ \varepsilon\ \sigma\ T_{atmosphere}^{4} = \varepsilon\ \sigma\ T_{ground}^{4}$$

or

$$T_{ground} = \sqrt[4]{2}\ T_{atmosphere}$$

The temperature of the ground must be warmer than the skin temperature, by a factor of the fourth root of two, an irrational but not transcendental number that equals about 1.189. The ground is warmer than the atmosphere by about 19%. When we do the calculation T_{ground} for Venus, Earth, and Mars in Table 1, we see that we are getting Earth about right, Mars too warm, and Venus not yet warm enough.

The blackbody atmospheric layer is not a source of energy, like some humungous heat lamp in the sky. How then does it change the temperature of the ground? I am going to share with you what is perhaps my favorite earth-sciences analogy, that of the equilibrium water level in a steadily filled and continuously draining sink. Water flowing into the sink, residing the in the sink for a while, and draining away is analogous to energy flowing into and out of the planet. Water drains faster as the level in the sink rises, as the pressure from the column of water pushes water down the drain. This is analogous to energy flowing away faster as the temperature of the planet increases, according to $\varepsilon\ \sigma\ T^4$. Eventually the water in the sink reaches a level where the outflow of water balances the inflow. That's the equilibrium value, and is analogous to the equilibrium temperature we calculated for the layer model. We constrict the drain somewhat but putting a penny down on the filter. For a while, the water drains out more slowly, and the water level in the sink rises because the water budget imbalance. The water level rises until the higher water level pushes water down the drain fast enough to balance the faucet again. A greenhouse gas, like the penny in the drain filter, makes it more difficult for the heat to escape the earth. The temperature of the earth rises until the fluxes balance again.

Take-Home Points

The outflow of IR energy from a planet must balance heating from the sun.

The planet accomplishes this act of energetic housekeeping by adjusting its temperature.

Absorption of outgoing IR light by the atmosphere warms the surface of the planet, as the planet strives to balance its energy budget.

Lab

1. The moon with no heat transport. The layer model assumes that the temperature of the body in space is all the same. This isn't really very accurate, as you know that it's colder at the poles than it is at the equator. For a bare rock with no atmosphere or ocean, like the moon, the situation is even worse, because fluids like air and water are how heat is carried around on the planet. So let's make the other extreme assumption, that there is no heat transport on a bare rock like the moon. What is the equilibrium temperature of the surface of the moon, on the equator, at local noon, when the sun is directly overhead? What is the equilibrium temperature on the dark side of the moon?

2. A two-layer model. Insert another atmospheric layer into the model, just like the first one. The layer is transparent to visible light but a blackbody for infrared.

a) Write the energy budgets for both atmospheric layers, for the ground, and for the earth as a whole, just like we did for the one-layer model.

b) Manipulate the budget for the earth as a whole to obtain the temperature T_2 of the top atmospheric layer, labeled Atmospheric Layer 2 in Figure 2-6. Does this part of the exercise seem familiar in any way? Does the term skin temperature ring any bells?

c) Insert the value you found for T_2 into the energy budget for layer 2, and solve for the temperature of layer 1 in terms of layer 2. How much bigger is T_1 than T_2?

d) Now insert the value you found for T_1 into the budget for atmospheric layer 1, to obtain the temperature of the ground, T_{ground}. Is the greenhouse effect stronger or weaker because of the second layer?

3. Nuclear Winter. Let's go back to the 1-layer model, but let's change it so that the atmospheric layer absorbs visible light rather than allowing to pass through (Figure 2-7). This could happen if the upper atmosphere were filled with dust. For simplicity, let's assume that the albedo of the earth remains the same, even though in the real world it might change with a dusty atmosphere. What is the temperature of the ground in this case?

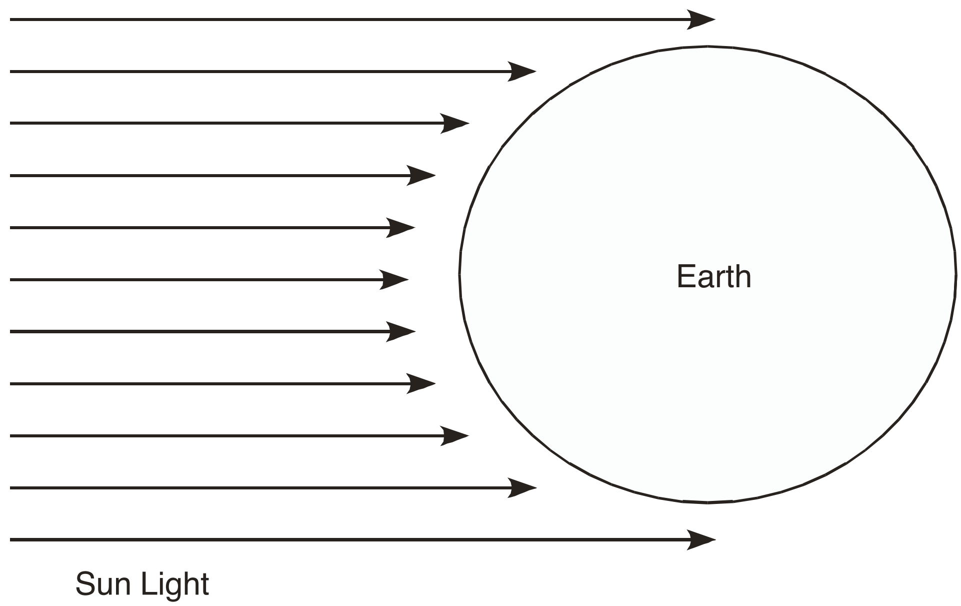

Figure 2-1

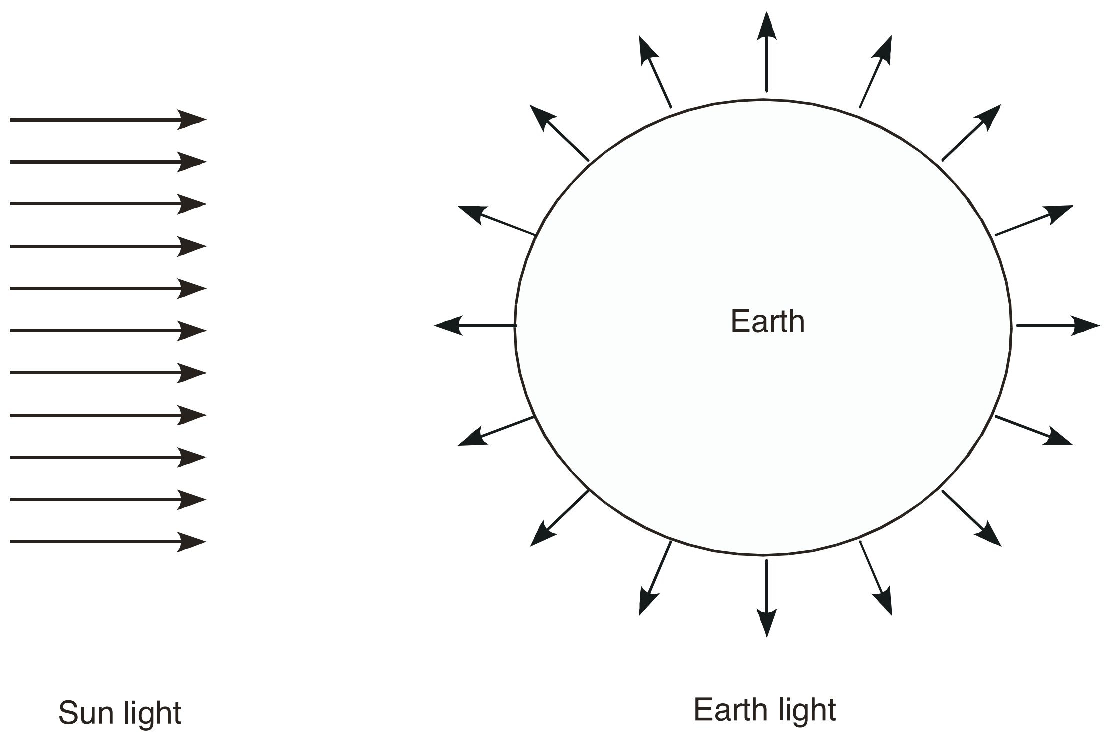

Figure 2-2

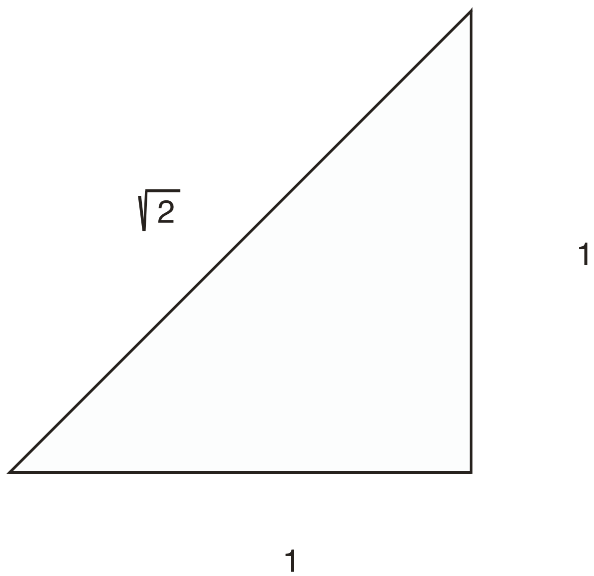

Figure 2-3

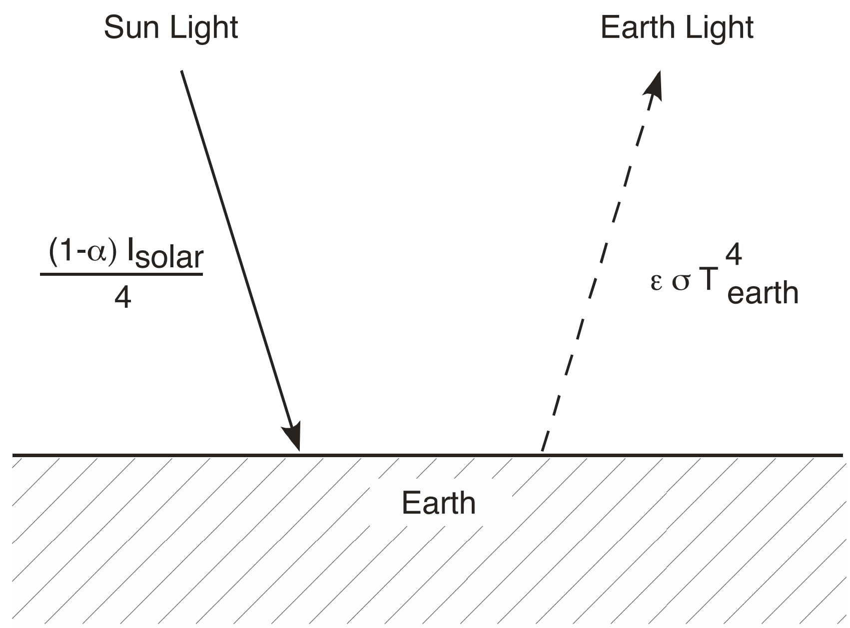

Figure 2-4

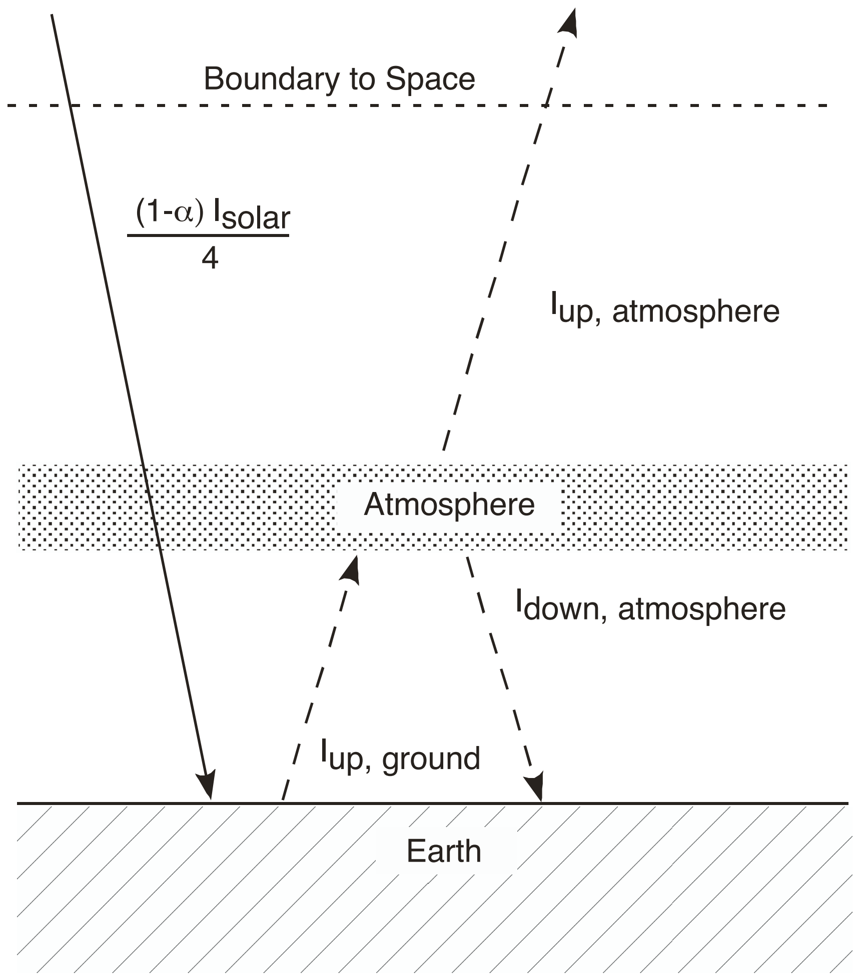
Boundary to Space
$\frac{(1-\alpha)\, I_{solar}}{4}$
$I_{up,\ atmosphere}$
Atmosphere
$I_{down,\ atmosphere}$
$I_{up,\ ground}$
Earth

Figure 2-5

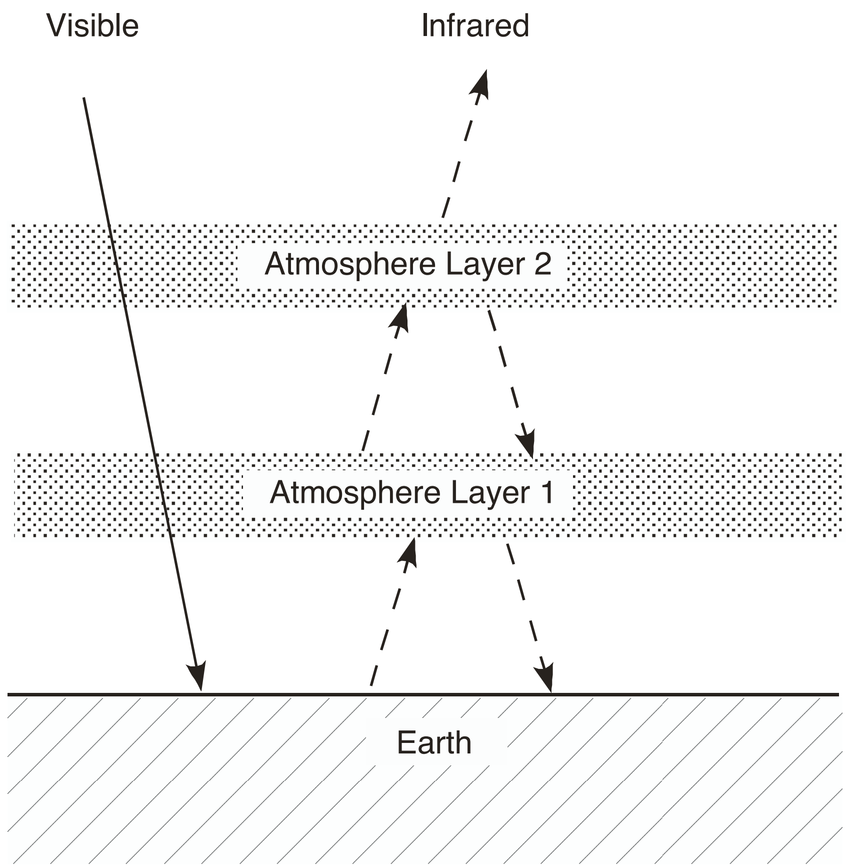
Visible
Infrared
Atmosphere Layer 2
Atmosphere Layer 1
Earth

Figure 2-6

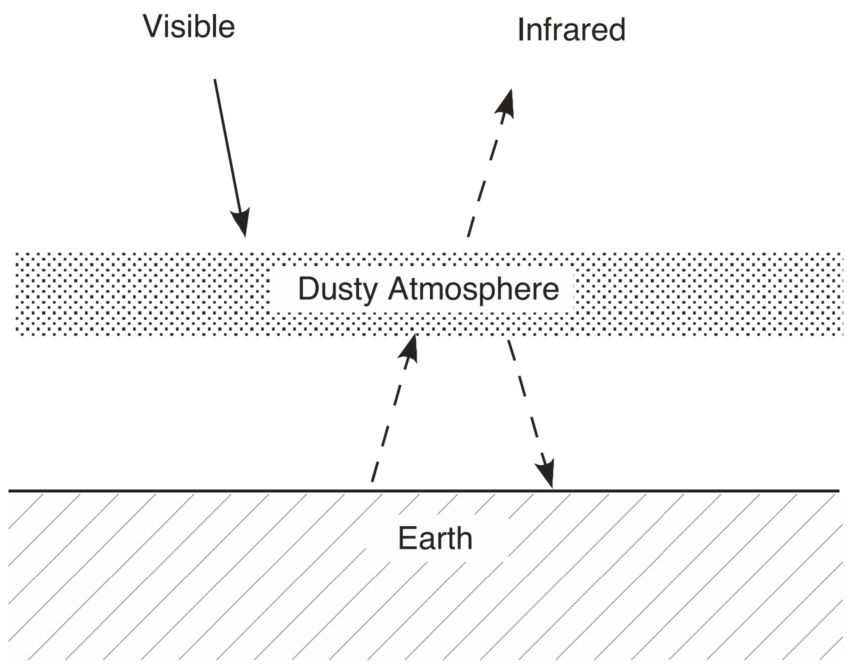

Figure 2-7