Vector and Geometric Calculus

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Geometry without algebra is dumb! - Algebra without geometry is blind!

- David Hestenes

The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn.

- Allan McRobie and Joan Lasenby

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Preface

Vector and Geometric Calculus is intended for the second year vector calculus course. It is a sequel to my text *Linear and Geometric Algebra.* That text is a prerequisite for this one. Single variable calculus is also a prerequisite.

Linear algebra and vector calculus have provided the basic vocabulary of mathematics in dimensions greater than one for the past one hundred years. Geometric algebra generalizes linear algebra in powerful ways. Similarly, geometric calculus generalizes vector calculus in powerful ways.

Traditional vector calculus topics are covered here, as they must be, since readers will encounter them in other texts and out in the world.

The final chapter is a brief introduction to (mostly 3D) differential geometry, used today in many disciplines, including architecture, computer graphics, computer vision, econometrics, engineering, geology, image processing, and physics.

Tensor calculus and differential forms are two formalisms extending vector calculus beyond three dimensions. Geometric calculus provides an at once simpler, more general, more powerful, and easier to grasp way to break loose from $\mathbb{R}^3$.[1] Section 5.4, *Exact Fields*, translates elementary differential forms definitions, theorems, and examples to geometric calculus.

Linear algebra is the natural mathematical background for vector calculus. Yet even today it is unusual for a vector calculus text to have a linear algebra prerequisite. This has to do, I suppose, with authors and publishers wanting to reach the largest possible audience. I cite my text *Linear and Geometric Algebra* freely and pervasively to advantage.

Vector and geometric algebra and differential vector and geometric calculus (Part II of this book) are excellent places to help students better understand and create proofs. But for integral calculus (Part III) rigorous proofs of fundamental theorems at the level of this book are mostly impossible. So I do not try.

Instead, I use the language of infinitesimals, while making it clear that they do not exist within the real number system. I believe that the first and most important way to understand integrals is intuitively: they "add infinitely many infinitesimal parts to give a whole". *This is how people who use calculus think.* Rigor can come later.

[1] D. Hestenes and G. Sobczyk have argued in detail the superiority of geometric calculus over differential forms: (*Clifford Algebra to Geometric Calculus*, D. Reidel, Dordrecht Holland 1984, Section 6.4, especially at the end.)

Others endorse this approach: "An approach based on [infinitesimals] closely reflects the way most scientists and engineers successfully use calculus. We continue to find it remarkable that the mainstream mathematics community insists on downplaying the use of infinitesimals, most especially when teaching calculus."[2] "The fact is that in many situations ... the interpretation of the integral as a sum of infinitesimals is the clearest way to understand what is going on."[3] From Lagrange: "When we have grasped the spirit of the infinitesimal method, and have verified the exactness of its results, ... we may employ infinitely small quantities as a sure and valuable means of shortening and simplifying our proofs.[4] And even Cauchy: "My main aim has been to reconcile the rigor which I have made a law in my Cours d'Analyse, with the simplicity that comes from the direct consideration of infinitely small quantities"[5]

There are over 200 exercises interspersed with the text. They are designed to test understanding of and/or give simple practice with a concept just introduced. My intent is that readers attempt them while reading the text. That way they immediately confront the concept and get feedback on their understanding. There are also more challenging problems at the end of most sections – almost 200 in all.

The exercises replace the "worked examples" common in most mathematical texts, which serve as "templates" for problems assigned to students. We teachers know that students often do not read the text. Instead, they solve assigned problems by looking for the closest template in the text, often without much understanding. My intent is that success with the exercises requires engaging the text.

Some exercises and problems require the use of the free multiplatform Python module $\mathcal{GA}$lgebra. It is based on the Python symbolic computer algebra library SymPy (Symbolic Python). GAlgebraPrimer.pdf describes the installation and use of the module. $\mathcal{GA}$lgebra is available at the book's web site.

With the June 2024 printing, $\mathcal{GA}$lgebra is no longer being updated. It still installs and runs, but unfortunately it is not in perfect shape. Please report any problems you have and I'll update GAlgebraPrimer.

Everyone has their own teaching style, so I would ordinarily not make suggestions about this. However, I believe that the unusual structure of this text (exercises instead of worked examples), requires an unusual approach to teaching from it. I have placed some thoughts about this in the file "VAGC Instructor.pdf" at the book's web site. Take it for what it is worth.

The first part of the index is a symbol index.

Some material which is difficult or less important is printed in this smaller font.

[2]Tevian Dray and Corinne Manogue, *Using Differentials to Bridge the Vector Calculus Gap*, The College Mathematics Journal **34**, 283-290 (2003).

[3]Gerald Folland, Advanced Calculus, p. 157, Pearson (2001).

[4]Méchanique Analytique, Preface; Ouveres, t. 2 (Paris, 1988), p. 14.

[5]Quoted in *Cauchy's Continuum*, Karin Katz and Mikhail Katz, Perspectives on Science **19**, 426-452. Also at arXiv:1108.4201v2.

There are several appendices. Appendix A reviews some parts of *Linear and Geometric Algebra* used in this book. Appendix B provides a list of some geometric calculus formulas from this book. Appendix C provides a short comparison of differential forms and geometric calculus. Appendix D proves a couple of technical results needed in the text.

Numbered references to theorems, figures, etc. preceded by "LAGA" are to *Linear and Geometric Algebra.*

There are several URL's in the text. To save you typing, I have put them in a file "URLs.txt" at the book's web site.

Please send corrections, typos, or any other comments about the book to me. I will post them on the book's web site as appropriate.

Acknowledgements. I thank Dr. Eric Chisolm, Greg Grunberg, Professor Philip Kuntz, James Murphy, and Professor John Synowiec for reading all/most of the text and providing and helpful comments and advice. Professor Mike Taylor answered several questions. I give special thanks to Greg Grunberg and James Murphy. Grunberg spotted many errors, made many valuable suggestions and is an eagle eyed proofreader. Murphy suggested major revisions in the ordering of my chapters.

I thank Dr. Isaac To and Dr. Nicholas R. Todd for pointing out errors.

I also thank the ever cooperative Alan Bromborsky for extending $\mathcal{GA}$lgebra to make it more useful to the readers of this book.

Thanks again to Professor Kate Martinson for help with the cover design.

> In general the position as regards all such new calculi is this - That one cannot accomplish by them anything that could not be accomplished without them. However, the advantage is that, provided such a calculus corresponds to the inmost nature of frequent needs, anyone who masters it thoroughly is able - without the unconscious inspiration of genius which no one can command - to solve the respective problems, indeed to solve them mechanically in complicated cases in which, without such aid, even genius becomes powerless. Such is the case with the invention of general algebra, with the differential calculus, ... Such conceptions unite, as it were, into an organic whole countless problems which otherwise would remain isolated and require for their separate solution more or less application of inventive genius.
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> - C. F. Gauss

Printings

From time to time I issue new printings of this book, with corrections and improvements. The printing version is shown on the title page.

Second Printing. I thank again Gregory Grunberg for many suggestions and expert proofreading. Christoph Bader and Dr. Gavin Polhemous pointed out shortcomings in the notation of Section 5.5. And I thank Dr. Manuel Reenders, a recent arrival, for many suggestions and corrections, especially with regard to the exercises and problems.

Third Printing. I thank a new eagle eyed reader, Nicholas H. Okamoto, for sending me errata.

Fifth Printing. I thank the very careful new reader Professor Mark R. Treuden for helpful comments and corrections.

August 2019 Printing. A new Section 5.4, *Exact Fields*, translates differential forms language to geometric calculus language: closed fields, exact fields potentials, etc.. It was gathered and improved from existing sections.

May 2020 Printing. The idea of a tangent map has been moved to a more appropriate place. There are a few new exercises/problems. All errors known to me have been corrected. All were minor.

October 2020 Printing. I've added material on the Helmholtz decomposition. The last part of Section 10.4 has been moved to Section 10.5. The section also contains some recently published results about antiderivatives. The new Section 11.4 introduces the differential geometry of manifolds of arbitrary dimension. There are several other small improvements.

January 2021 Printing. There are minor improvements.

June 2021 Printing. Eq. (9.3) is new. It provides a better understanding of the definitions of the surface and flux integrals. The last two sections of Chapter 10 have been rearranged. A geometric calculus version of the Helmholtz decomposition has been added to Section 10.5 to go with the vector calculus version in Section 5.4. There is a new Section 11.4, Manifolds in $\mathbb{R}^n$. There are many minor improvements and corrections.

January 2022 Printing. I have tried to make the text clearer in a number of places. Section 10.5, Analytic Functions, has been rearranged yet again. All errors/typos known to me have been corrected.

The text is improved in several places. The definition of limit have been given a new pictorial form, enabling better understanding of this concept. All errors/typos known to me have been corrected.

September 2023 Printing. There is a new short description of the gradient descent algorithm. All errors/typos known to me have been corrected.

December 2023 Printing. I have fixed many errors in citations to *Linear and Geometric Algebra.* All errors/typos known to me have been corrected.

June 2024 Printing. All errors/typos known to me have been corrected.

April 2025 Printing. All errors/typos known to me have been corrected.

To the Reader

Appendix A is a review of some items from *Linear and Geometric Algebra* (LAGA) used in this book. A quick read through it might be helpful before starting this book.

I repeat here my advice from *Linear and Geometric Algebra*.

Research clearly shows that *actively* engaging course material improves learning and retention. Here are some ways to actively engage the material in this book:

- Don't just read the text, *study* the text. This may not be your habit, but many parts of this book require reading and rereading and rereading again later before you will understand.

- Instructors in your previous mathematics courses have probably urged you to try to *understand*, rather than simply memorize. That advice is especially appropriate for this text.

- Many statements in the text require some thinking on your part to understand. Take the time to do this instead of simply moving on. Sometimes this involves a small computation, so have paper and pencil on hand while you read.

- Definitions are important. Take the time to understand them. You cannot know a foreign language if you do not know the meaning of its words. So too with mathematics. You cannot know an area of mathematics if you do not know the meaning of its defined concepts.

- Theorems are important. Take the time to understand them. If you do not understand what a theorem says, then you cannot understand its applications.

- Exercises are important. Attempt them as you encounter them in the text. They are designed to test your understanding of what you have just read. Some are trivial, there just to make sure that you are paying attention. But do not expect to solve them all. Even if you cannot solve an exercise you have learned something: you have something to learn!

The exercises require you to think about what you have just read, think more, perhaps, than you are used to when reading a mathematics text. This is part of my attempt to help you start to acquire that "mathematical frame of mind".

Write your solutions neatly in clear correct English.

- Proofs are important, but perhaps less so than the above. On a first reading, don't get bogged down in a difficult proof. On the other hand, one goal of this course is for you to learn to read and construct mathematical proofs better. So go back to those difficult proofs later and try to understand them.

- Important: take the above points seriously!

The World Wide Web makes it possible for me to leave out material that I would otherwise have to include. For example, the book refers to the *Coulomb force* without defining it. Perhaps you already know what it is. If not, and you want to know, actively engage the course material: Google it.

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