Elementary Discussion of the Geometric Phase

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1. ρ_1 is *in phase with* ρ_0 if $\langle \rho_0 \,|\, \rho_1 \rangle > 0$.

2. Given ρ_1 and ρ_0 with $\langle \rho_0 \,|\, \rho_1 \rangle \neq 0$, there is a unique $\alpha \,(\mathrm{mod} 2\pi)$ so that $\mathrm{e}^{\mathrm{i}\alpha} |\, \rho_1 \,\rangle$ is in phase with $|\, \rho_0 \,\rangle$: $\alpha = -\arg\left(\langle \rho_0 \,|\, \rho_1 \rangle\right)$.

3. The α of §2 maximizes $\| |\, \rho_1 \,\rangle + \mathrm{e}^{\mathrm{i}\beta} |\, \rho_0 \,\rangle \|^2 = 2 + 2\mathrm{Re}\langle \rho_0 \,|\, \mathrm{e}^{\mathrm{i}\beta} \rho_1 \rangle$, $(0 \leq \beta < 2\pi)$. This motivates "in phase with".

4. The relation "in phase with" is not transitive: if $|\, \rho_2 \,\rangle$ is in phase with $|\, \rho_1 \,\rangle$, and $|\, \rho_1 \,\rangle$ is in phase with $|\, \rho_0 \,\rangle$, then $|\, \rho_2 \,\rangle$ need not be in phase with $|\, \rho_0 \,\rangle$.

5. Let a differentiable curve of unit vectors $|\, \psi(t) \,\rangle, 0 \leq t \leq a$ be given.

6. Define $|\, \rho(t) \,\rangle = \mathrm{e}^{\mathrm{i}\alpha(t)} |\, \psi(t) \,\rangle$ so that (i) $|\, \rho(0 \,\rangle = |\, \psi(0) \,\rangle$ and (ii) $|\, \rho(t) \,\rangle$ "is locally in phase": $|\, \rho(t+dt) \,\rangle$ is in phase with $|\, \rho(t) \,\rangle$ for all $t, 0 \leq t < a$. From §4, $|\, \rho(a) \,\rangle$ need not be in phase with $|\, \rho(0) \,\rangle$.

7. Now choose γ by §2 so that $\mathrm{e}^{\mathrm{i}\gamma} |\, \rho(a) \,\rangle$ is in phase with $|\, \rho(0) \,\rangle$. γ is the *geometric phase* of the curve $|\, \psi(t) \,\rangle, 0 \leq t \leq a$.

8. γ is *geometric*: it is independent of the phases and parametrization of $|\, \psi(t) \,\rangle$. For according to §2, $\alpha(t)$, and thus γ, are uniquely determined by the locally in phase property of $|\, \rho(t) \,\rangle$.

9. The locally in phase condition for $|\, \rho(t) \,\rangle$ is $\alpha' = \langle \psi \,|\, \mathrm{i}\psi' \rangle$ $(= \langle \psi \,|\, \mathrm{H}\psi \rangle$ if Schrödinger's equation generates the curve). For we must have

$$\langle \rho(t) \,|\, \rho(t+dt) \rangle = 1 + \{\mathrm{i}\alpha'(t) + \langle \psi(t) \,|\, \mathrm{i}\psi'(t) \rangle\}\, dt > 0.$$

(α' is real: differentiating $\langle \psi \,|\, \psi \rangle = 1$ shows that $\mathrm{Re}\langle \psi \,|\, \psi' \rangle = 0$.)

10. We evaluate γ. First, $\alpha(a) = \int_0^a \alpha'(t)dt = \int_0^a \langle \psi(t) \,|\, \mathrm{i}\psi'(t) \rangle dt$. Thus

$$\begin{aligned} \gamma &= -\arg\langle \rho(0) \,|\, \rho(a) \rangle \\ &= -\arg\langle \psi(0) \,|\, \mathrm{e}^{\mathrm{i}\alpha(a)} \psi(a) \rangle \\ &= -\arg\langle \psi(0) \,|\, \psi(a) \rangle - \int_0^a \langle \psi(t) \,|\, \mathrm{i}\psi'(t) \rangle dt. \qquad (1) \end{aligned}$$

11. Consider a simple closed (up to phase) curve of states of a spin-$\frac{1}{2}$ particle. Since each state is spin up in some direction, the curve corresponds to a closed curve C on the unit sphere. Let C bound region A, which subtends solid angle Ω. Let $|\,\psi(\theta,\phi)\,\rangle = \cos(\frac{\theta}{2})|+z\,\rangle + e^{i\phi}\sin(\frac{\theta}{2})|-z\,\rangle$ represent spin up in the direction (θ,ϕ). Apply Eq. (1) to the closed curve C, compute $\langle\psi\,|\,\psi\,'\rangle = \langle\psi\,|\,\psi_\theta\,\theta\,' + \psi_\phi\,\phi\,'\rangle$, apply Green's theorem, and use the element of surface area on the unit sphere, $(\sin\theta\,d\theta)d\phi$:

$$\gamma = -\int_0^a \langle\psi\,|\,i\psi'\rangle dt = \oint_C \sin^2\left(\frac{\theta}{2}\right)d\phi = \frac{1}{2}\int\!\!\int_A d\theta\sin\theta\,d\phi = \frac{\Omega}{2}.$$