World's Fastest Derivation of the Lorentz Transformation[1]

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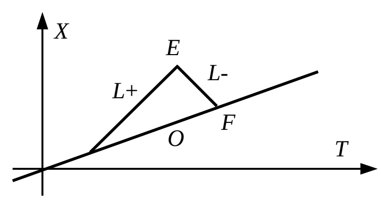
Assume: (A) The speed of light is the same in all inertial frames. (Take $c = 1$.) (B) Inertial frames are homogeneous and spatially isotropic.[3]

Let an inertial clock move with speed v in an inertial frame I. Let $\tau = \tau(t)$ be the reading of the clock. By (B), $d\tau/dt$ is the same for all inertial clocks with speed v in I. Then by the relativity principle, $d\tau/dt$ is the same for inertial clocks with speed v in other inertial frames. Set $\gamma = \gamma(v) = d\tau/dt$.[4]

In the figure, E is an arbitrary event in I, $L+$ and $L-$ are the two light worldlines through E, and O is the worldline of the spatial origin of an inertial frame I' moving with velocity v in I. On O,



$$X' = 0,\ X = vT,\ \text{and } T = \gamma T'.$$

Thus on O,

$$T + X = \gamma(1 + v)(T' + X') \quad (1)$$
$$T - X = \gamma(1 - v)(T' - X'). \quad (2)$$

Since $c = 1$ in I, an increase in T along $L-$ is accompanied by an equal decrease in X. Thus $T + X$ is the same at E and F. Likewise, since $c = 1$ in I', $T' + X'$ is the same at E and F. Thus Eq. (1), which is true at F, is also true at E. Similar reasoning using $L+$ proves Eq. (2) true at E. Add and subtract Eqs. (1) and (2):

$$T = \gamma(T' + vX') \quad (3)$$
$$X = \gamma(vT' + X'). \quad (4)$$

For $X = 0$ in Eq. (4), $X' = -vT'$; the origin of I has velocity $-v$ in I'.[5] Thus, switching I and I' and using (B), the reasoning for Eq. (1) also gives $T' + X' = \gamma(1 - v)(T + X)$. Substituting this in Eq. (1) gives $\gamma = (1 - v^2)^{-\frac{1}{2}}$.

Equations (3) and (4), with $\gamma = (1 - v^2)^{-\frac{1}{2}}$ are the Lorentz transformations at the arbitrary event E.

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[1] Improved from Am. J. Phys. **49** 493 (1981), with a new title.

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[3] We do not assume that the Lorentz transformation is linear.

[4] This does not assume time dilation: as far as (B) is concerned, γ could be identically 1.

[5] This *reciprocity principle*, just proved, is often assumed.